Notes on analytic de Rham stacks

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My personal notes from workshop at IMPAN on [Cam24b]. I have left out some arguments I found uninteresting and available in [Cam24b]. Any mistakes in these notes are due to me. Some missing parts are left in **bold face text**. If you spot any typos, mistakes, or know the answer to my questions, please do not hesitate to email me!

Contents

1 Adic spaces as categorified locales (Akhil Mathew)

Setup :

- A Tate ring, i.e. topological ring defined via a non-Archimedean Banach norm and exists $\pi \in A^\times$ with $|\pi| < 1$.
- A^+ subring of A such that
  1. $A^+ \subseteq A^\circ$
  2. A^+ open
  3. A^+ integrally closed

Huber defines a spectral space $\mathrm{Spa}\,(A, A^+)$ and presheaves $\mathcal{O}, \mathcal{O}^+$ which are sheaves in favorable situations.

Goal : recover Huber's construction from a categorical generalisation.

> **Definition**
>
> Let $\mathcal{C} \in \mathrm{cAlg}(\mathrm{Pr}^L_{st})$. [a] For A commutative algebra in $\mathcal{C}$, A is *idempotent* when $A \otimes A \xrightarrow{\sim} A$.
>
> ---
>
> [a] Presentable stable infinity categories with colimit preserving functors. This has symmetric monoidal structure by the Lurie tensor product.

Remark. 1. $\mathrm{Mod}_A(\mathcal{C})$ is a full subcategory of $\mathcal{C}$.

2. Collection of idempotent algebras forms a poset with all colimits. [Aok23, Cor.2.8.] In particular, it is small. (This comes from presentability of $\mathcal{C}$.) [1]

> **Definition**
>
> A morphism $i^* : \mathcal{C} \to \mathcal{D}$ in $\mathrm{cAlg}(\mathrm{Pr}^L_{st})$ is a *closed immersion* when the right adjoint i_* is fully faithful and satisfies projection formula.[a]
>
> ---
>
> [a] Right adjoint exists by adjoint functor theorem for presentably infinity categories. Indeed, this is one of the reasons for restricting to Pr^L_{st}.

Examples :

1. $\_ \otimes A : \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ is a closed immersion.
2. X topological, $Z \subseteq X$ closed subspace. Then $i^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ is a closed immersion with idempotent algebra i_*1.
3. Let X qcqs scheme and $U \subseteq X$ open subscheme. Then restriction $j^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$ is a closed immersion with $Rj_*\mathcal{O}_U$ is the corresponding idempotent algebra. [2]

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[1] Mapping space between any two is empty or contractible.

[2] Remark from audience : The complement of U in X is a formal neighbourhood, which we should think of as a tubular neighbourhood, in particular *open*.

Remark. If X Noetherian, then the category of idempotent algebras in $\mathrm{QCoh}(X)$ is equivalent to the ind-completion of the opposite of the Zariski topology of X.

Another example : R commutative ring, $I \subseteq R$ flat with $I^2 = I$ then $R/I \in D(R)$ is idempotent.

Example for these talks : in $D((\mathbb{Z}[T], \mathbb{Z})_\square)$, the algebras $\mathbb{Z}((1/T))$ and $\mathbb{Z}[[T]]$ are idempotent. In general, for finitely generated R with an ideal I, then $R_I^\wedge$ gives an idempotent algebra in $D((\mathbb{Z}[T], \mathbb{Z})_\square)$.

Example : $D((\mathbb{Q}_p[T], \mathbb{Z})_\square)$ the derived category of modules of $\mathbb{Q}_p[T]$ in $\mathbb{Z}_\square$-modules. Then $\mathbb{Q}_p\langle T\rangle$ is idempotent. Also the algebra of overconvergent functions $\mathbb{Q}_p\left(T\right)^\dagger$.

> **Proposition**
>
> Given A idempotent algebra in $\mathcal{C} \in \mathrm{cAlg}(\mathrm{Pr}^L_{st})$, consider Bousfield localisation
>
> $$\mathcal{C}/\langle A\rangle := \{X \in \mathcal{C} \text{ s.t. } \mathrm{Hom}(A, X) = 0\}$$
>
> Let $j_* : \mathcal{C}/\langle A\rangle \to \mathcal{C}$ be be inclusion of the full subcategory. Then :
>
> - j_* has a left adjoint $j^* : \mathcal{C} \to \mathcal{C}/\langle A\rangle$.
> - j^* has a further left adjoint $j_!$ maps X to $\mathrm{Fib}(1 \to A) \otimes X$.
>
> In general for $j^* : \mathcal{C} \to \mathcal{D}$, we call is an open immersion when it has a left adjoint $j_!$ which is fully faithful and satisties projection formula.

Example : For X qcqs scheme and $U \subseteq X$ open and $Z \subseteq X$ closed in the complement, then $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X_Z^\wedge)$ is an open immersion. [1]

Example : X topological and $Z \subseteq X$ closed with complement U, then $j^* : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ is an open immersion.

Example : (R, I) almost setup, then $D(R) \to D^a(R)$ is open immersion.

Example : The tensor functor $D((\mathbb{Z}[T], \mathbb{Z})_\square) \to D(\mathbb{Z}[T]_\square)$ is the complement open immersion to $\mathbb{Z}((1/T))$. [Cam24a, Prop.5.2.2.]

Example : The tensor functor $D((\mathbb{Z}[T], \mathbb{Z})_\square) \to D((\mathbb{Z}[T, 1/T], \mathbb{Z}[1/T])_\square)$ is the open immersion complement to $\mathbb{Z}[[T]]$.

> **Proposition – Balman-Krause-Stevensen**
>
> For $\mathcal{C} \in \mathrm{cAlg}(\mathrm{Pr}^L_{st})$, the infinity category $\mathrm{Idem}(\mathcal{C})$ is a locale. Denote by $\mathrm{Spec}^{\mathrm{big}}(\mathcal{C})$.

> **Definition**
>
> A poset $\mathcal{A}$ is a locale when :
>
> 1. any subset has supremum, i.e. has all colimits
> 2. $X \wedge \bigvee_{i\in I} Y_i = \bigvee_{i\in I}(X \wedge Y_i)$.
>
> A morphism $\mathcal{A} \to \mathcal{B}$ is a morphism of partially ordered sets $\mathcal{B} \to \mathcal{A}$ preserving all colimits and finite

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[1] There exists finitely generated ideal of definition for Z by Noetherian assumptions.

> limits. [a]
>
> ---
>
> [a] Remark from audience : To explain why the morphisms go backwards, maybe it is easier to say these are 0-topoi.

Example : X topological space, $\mathrm{Open}(X)$ gives a locale. Then given $f : X \to Y$ of topological spaces, we get $f^* : \mathrm{Open}(Y) \to \mathrm{Open}(X)$, whicch we flip around $f_* : \mathrm{Open}(X) \to \mathrm{Open}(Y)$ to get a morphism of locales. Akhil goes on to give the usual story about locales and sober spaces, locales with enough points (spatial locales), etc. Point is locales generalise topological spaces. Locales fully faithfully embed into ∞-topoi. The topoi which come from locales, *localic*, are precisely the ones generated under colimits by subobjects (of the final object? Akhil didn't say but that makes sense).

> **Definition**
>
> Given a locale $\mathcal{A}$ we have a notion of a sheaf on it.
>
> For $\mathcal{C} \in \mathrm{cAlg}(\mathrm{Pr}^L_{st})$, there is a sheaf on $\mathrm{Spec}^{\mathrm{big}}(\mathcal{C})$ given by taking $A \in \mathrm{Idem}(\mathcal{C})$ to $\mathcal{C}/\langle A \rangle$.

Remark. We don't know what $\mathrm{Spec}^{\mathrm{big}}(\mathcal{C})$ looks like. Even for $\mathcal{C} = D(R)$, we do not know if it is spatial.

Remark. $\mathrm{Spec}^{\mathrm{big}}\,\mathcal{C}$ does not determine $\mathcal{C}$. Take $\mathcal{C} = D(K)$ where K is a field.

> **Definition**
>
> A categorified locale consists of
>
> 1. X is a locale
> 2. $\mathcal{C} \in \mathrm{cAlg}(\mathrm{Pr}^L_{st})$
> 3. $f : \mathrm{Spec}^{\mathrm{big}}\,\mathcal{C} \to X$ a morphism of locales.
>
> A morphism of categorified locales is what you think it is. [Cam24b, Def.2.2.6.]

Example : (A, A^+) be a Tate-Huber pair. Let $\mathcal{C} := D((A, A^+)_\square)$. For each $f \in A(*)$, we have $\mathbb{Z}[T] \to A, T \mapsto f$. Then using the morphism of tensor categories

$$D((\mathbb{Z}[T], \mathbb{Z})_\square) \xrightarrow{\otimes} D((A, A^+)_\square)$$

we get idempotent algebras in $D((A, A^+)_\square)$.

$$A \otimes^\square_{\mathbb{Z}[T]} \mathbb{Z}((1/T)) \qquad\qquad A \otimes^\square_{\mathbb{Z}[T]} \mathbb{Z}[[T]]$$

Take the subframe generated by all of these.

Observation : $\mathbb{Z}((1/T)), \mathbb{Z}[[T]]$ are compact in solid modules over $\mathbb{Z}[T]$, which implies they are compact as idempotent algebras. This implies the subframe/locale they generate defines a spectral space. There is a duality between spectral spaces and distributive lattice generated under compact objects stable under finite limits.

> **Proposition**

> The above subframe is precisely $\mathrm{Spa}\,(A, A^+)$. For $\mathbb{Z}[T] \to A, T \mapsto f \in A(*)$,
>
> $$A \otimes^{\square}_{\mathbb{Z}[T]} \mathbb{Z}((1/T)) \rightsquigarrow |f| \leq 1$$
> $$A \otimes^{\square}_{\mathbb{Z}[T]} \mathbb{Z}[[T]] \rightsquigarrow |f| \geq 1$$

Proof. Need to calculate points in the locale. A point is a locale morphism $\{0, 1\} \to \mathrm{Spec}^{\mathrm{big}}\, \mathcal{C}$. One can use this to define a valuation. The purpose of a valuation is to know for $f, g \in A$ whether $|f| \leq |g|$ or $|g| \leq |f|$. So first restrict to $g \neq 0$ then "ask yes or no" for $|f/g| \leq 1$. $\square$

Can do with usual modules and get opposite of Zariski spectrum. Berkovich spectrum can also be recovered for by using overconvergent algebras.

2 Dagger-nilradicals and bounded affinoid rings I (Emanuel Reinecke)

Goal of workshop : For X rigid space over $\mathbb{Q}_p$, we want to associate an "analytic stack"

$$A \text{ bounded affinoid ring} \mapsto X(A^{\dagger\mathrm{red}})$$

where $A^{\dagger\mathrm{red}} = A/\mathrm{Nil}^{\dagger}(A)$.

Setup :

- work in the ∞-category of animated $\mathbb{Z}_{\square}$-algebras $\mathrm{cAlg}(D(\mathbb{Z}_{\square}))$.
- Fix $(R, R^+) := (\mathbb{Z}((\pi)), \mathbb{Z}[[\pi]])$ and $R_{\square} := (R, R^+)_{\square}$. Intuitively, this takes π-completion.

First, topologically nilpotent and power bounded elements.

Idea : For A animated $\mathbb{Z}_{\square}$-algebra, then as the underlying condensed ring should satisfy for S extremally disconnected, $A^{\circ\circ}(S) := \{f : S \to A \text{ s.t. } \forall s_1, s_2, \dots \in S, \lim_{n\to\infty} f(s_1)\cdots f(s_n) = 0\}$.

For A animated $R_{\square}$-algebra, $A^{\circ}(S) := \{f : S \to A \text{ s.t. } \forall s_1, s_2 \in S, \{s_1, s_2\} \text{ bounded}\}$.

We have an adjunction :

$$\begin{tikzcd}
\bigoplus_{n\geq 0} \mathrm{Sym}^n \mathbb{Z}[S] & \mathbb{Z}[S] & S \text{ ext.dis.} \\
\mathrm{Cond(AniRing)} \arrow[r, shift right, "\mathrm{forget}"'] \arrow[r, phantom, "\perp"] & D_{\geq 0}(\mathbb{Z}) \arrow[l, shift right, "\mathrm{Sym}"']
\end{tikzcd}$$

where $\mathrm{Sym}^n \mathbb{Z}[S] := (\mathbb{Z}S^{\oplus n})_{\mathfrak{S}_n} = \mathbb{Z}[S^n_{\mathfrak{S}_n}] = \mathbb{N}[S]_{=n}$. Here $S^n_{\mathfrak{S}_n} := S^n/\mathfrak{S}_n$ quotient in the topos of condensed sets. Also $\mathbb{N}[S]_{=n}$ = continuous maps to discrete $\mathbb{N}$ such that the image sums to n.

Can do the same for $\mathcal{A} = \mathbb{Z}_{\square}, R_{\square}$.

Example : For S profinite, we have $C(S,\mathbb{Z}) = \bigoplus_{i\in I}\mathbb{Z}e_i$. (in notes on condensed sets). Then $\mathbb{Z}_\square[S] = \prod_{i\in I}\mathbb{Z}e_i^\vee$. This leads to

$$\mathrm{Sym}^n_{\mathbb{Z}_\square}\mathbb{Z}_\square[S] = \prod_{\underline{\alpha}\in I^n/\mathfrak{S}_n}\mathbb{Z}(e^\vee_{\leq\underline{\alpha}})$$

Definition

Let S profinite set.

1. $\mathbb{Z}_\square[[\mathbb{N}[S]]] := \varprojlim_{n\geq 0}(\mathbb{Z}_\square[\mathbb{N}[S]]/\mathrm{Sym}^n\mathbb{Z}_\square[S])$
2. $R^+_\square\langle\mathbb{N}[S]\rangle := \varprojlim_{n\geq 0}(R^+/\pi^n)_\square[\mathbb{N}[S]]$. Note that the $(R^+/\pi^n)_\square[\mathbb{N}[S]]$ are symmetric algebras over $(R^+/\pi^n)_\square$ on S.
3. $R_\square\langle\mathbb{N}[S]\rangle := R^+_\square\langle\mathbb{N}[S]\rangle[1/\pi]$.

Definition 1. For A animated $\mathbb{Z}_\square$-algebra,

$$A^{\circ\circ} : S \text{ ext.dis.} \mapsto \mathrm{Map}_{\mathrm{AniAlg}_{\mathbb{Z}_\square}}(\mathbb{Z}_\square[[\mathbb{N}[S]]], A)$$

Note $\mathbb{Z}_\square[\mathbb{N}[S]] \to \mathbb{Z}_\square[[\mathbb{N}[S]]]$ epi implies we have monomorphism of condensed sets $A^{\circ\circ} \to A$.

2. For A animated $R_\square$ algebra,

$$A^{\circ} : S \text{ ext.dis.} \mapsto \mathrm{Map}_{\mathrm{AniAlg}_{R_\square}}(R_\square\langle[\mathbb{N}[S]]\rangle, A)$$

Fact : $\mathbb{Z}_\square[[\mathbb{N}[S]]]$ and $R_\square\langle\mathbb{N}[S]\rangle$ are idempotent algebras over $\mathbb{Z}_\square[\mathbb{N}[S]]$ and $R_\square[\mathbb{N}[S]]$. This implies $A^{\circ\circ}, A^\circ$ are full condensed subanima of A. i.e. in condensed anima we have

$$\begin{tikzcd}
A^\circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
\pi_0 A^\circ \arrow[r] & \pi_0 A
\end{tikzcd}$$

and similarly for $A^{\circ\circ}$.

Consequently $\mathbb{Z}_\square[[\mathbb{N}[S]]]$ and $R_\square\langle\mathbb{N}[S]\rangle$ have cocommutative comultiplication from $s \mapsto s\otimes 1 + 1\otimes s$.

Example : Given A Tate-Huber ring, then choosing a pseudo-uniformizer gives a condensed $R_\square$-algebra. Then $A^{\circ\circ}, A^\circ$ gives the classical notions, independent of the choice of pseudo-uniformizer. At the level of elements, $A^\circ * (*) \subseteq A(*)$ is the subset of $R[T] \to A$ that extends along $R[T] \to R\langle T\rangle$. Similarly for $A^{\circ\circ}$ and along $\mathbb{Z}[T] \to \mathbb{Z}[[T]]$.

Question from audience : the extension data or property? Answer : Should be property because of $A^{\circ\circ}, A^\circ$ being full condensed subanima of A.

> **Definition**
>
> Let A be an animated $R_{\square}$-algebra,
>
> 1. $A^b := A^{\circ}[1/\pi]$ is the condensed animated subring of bounded elements of A.
> 2. A is bounded when $A^b \to A$ is an equivalence.

Example : Let A_0 be a π-adically complete animated $R^+_{\square}$-algebra. Then $A = A_0[1/\pi]$ is bounded.

Proof. WLOG A is π-torsion-free. STS we have extensions

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & A_0 \arrow[r, phantom, "\subseteq"] & A \\
R^+[\mathbb{N}[S]] \arrow[d] \arrow[ru, dashed] & & \\
R^+\langle \mathbb{N}[S] \rangle \arrow[d] \arrow[ruu, dashed] & & \\
R\langle \mathbb{N}[S] \rangle \arrow[ruuu, dashed] & &
\end{tikzcd}$$

The first exists by π-adic completeness of A_0. ☐

Example : A Tate-Huber. Then $A = A_0[1/\pi]$ for some π-adically complete ring of definition A_0. Then A is bounded as a $R_{\square}$-algebra using π as choice of pseudo-uniformizer.

Example : $R_{\square}[T]$ not bounded.

$$\begin{tikzcd}
R_{\square}[T] \arrow[r, "T \mapsto \pi^n T"] \arrow[d] & R_{\square}[T] \\
R_{\square}\langle T \rangle \arrow[ru, dashed, "\nexists"'] &
\end{tikzcd}$$

Example (not in original talk, but I find quite helpful) : Classically, for affinoids A over $\mathbb{Q}_p$, we have $A = \bigcup_{n \geq 0} p^{-n} A^{\circ}$ which is indeed $A^{\circ}[1/p]$. We can rephrase this in terms of module maps. That is, for every $f \in A$, which we can reinterpret as a $\mathbb{Q}_p$-vector space map, then there exists $n \geq 0$ such that we have a factoring

$$\begin{tikzcd}
\mathbb{Q}_p \arrow[r, "1\mapsto f"] \arrow[d, "p^n"'] & A \arrow[d, "p^n"] \\
\mathbb{Q}_p \arrow[r, "1\mapsto f"] \arrow[d, "1\mapsto T"'] & A \\
\mathbb{Q}_p\langle T\rangle \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

where the top square is in $\mathbb{Q}_p$-vector spaces and the dashed diagonal is a morphism of affinoids over $\mathbb{Q}_p$. The following criterion (2) is the generalisation of this to families of elements parameterised by a profinite set.

Proposition – Criteria of boundedness

Let A animated $R_\square$-algebra. Then

1. A bounded iff $\pi_0 A$ bounded.
2. A bounded iff there exists S_i profinite and $\bigoplus_i R_\square[S_i] \to A$ such that
   (a) surjective on π_0
   (b) $(A = A^\circ[1/\pi])$ for all i there exists n such that we have a factoring :

$$\begin{tikzcd}
R_\square[S_i] \arrow[r] \arrow[d, "\pi^n"'] & A \arrow[d, "\pi^n"] \\
R_\square[S_i] \arrow[r] \arrow[d] & A \\
R_\square\langle\mathbb{N}[S]\rangle \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Note that the square is happening in $D(R_\square)$ and the factoring we require is a morphism of animated condensed $R_\square$-algebra.

Proposition

The infinity category of bounded animated $R_\square$-algebras admits all small colimits and small limits.

1. colimits are computed in the infinity category of animated $R_\square$-algebras
2. limit of a system A_i is

$$\left(\varprojlim_{i\in I} A_i^\circ\right)[1/\pi]$$

where the limit is in animated condensed R°-algebras. [a]

---

[a] The example of increasing union of closed disks gives the interpretation as bounded functions.

Proof. Idea for tensor products : Take $\pi_0(B \otimes_A C)$ where $B \leftarrow A \rightarrow C$ in bounded animated $R_\square$-algebras. Then use criterion (2). $\square$

3 Dagger-nilradicals and bounded affinoid rings II (Emanuel Reinecke)

Idea for †-nilradical : For A animated $R_\square$-algebra, then $\mathrm{Nil}^\dagger(A)$ should be set of $a \in A(*)$ such that for all $n \geq 0$, $\pi^{-n}a$ is power bounded.

> **Definition**
>
> For S profinite set, define $R_\square\left\{\mathbb{N}[S]\right\}^\dagger$ as the colimit of
>
> $$R_\square\langle\mathbb{N}[S]\rangle \xrightarrow{\pi} R_\square\langle\mathbb{N}[S]\rangle \xrightarrow{\pi} \cdots$$

Note to self : intuitively, intersection of closed disks of radius $|\pi|^n$ as $n \to \infty$.

Remark.

1. has cocomutative comultiplication induced by $s \mapsto 1 \otimes s + s \otimes 1$.
2. Exists $R_\square\left\{\mathbb{N}[S]\right\}^\dagger \to R_\square\langle\mathbb{N}[S]\rangle \otimes_{R_\square} R_\square\left\{\mathbb{N}[S]\right\}^\dagger$ with $s/\pi^n \mapsto s \otimes (s/\pi^n)$. Note : these are all static.
3. $R_\square\left\{\mathbb{N}[S]\right\}^\dagger$ idempotent $R_\square[\mathbb{N}[S]]$-algebra.

> **Definition**
>
> For A animated $R_\square$-algebra,
>
> $$\mathrm{Nil}^\dagger(A) : S \text{ ext.dis.} \mapsto \mathrm{Map}_{\mathrm{AniAlg}_{R_\square}}(R_\square\left\{\mathbb{N}[S]\right\}^\dagger, A)$$

> **Proposition**
>
> (3) implies $\mathrm{Nil}^\dagger(A)$ is a full condensed subanima of A. (1) and (2) together imply it is full condensed A^b-ideal. [a]
>
> ---
>
> [a] It was not obvious to me what an ideal was in the animated setting. Thankfully, someone provided a reference. [Mao24]

> **Definition**
>
> Let A be a bounded animated $R_\square$-algebra.
>
> 1. $A^{\dagger\mathrm{red}} := \mathrm{CoFib}(\mathrm{Nil}^\dagger(A) \to A)$ in $D(A)$.
> 2. A is called reduced when $A \to A^{\dagger\mathrm{red}}$ is equivalence.

> **Proposition**
>
> $A^{\dagger\mathrm{red}}$ is static and thus has an obvious ring structure.

Proof. $\mathrm{Nil}^\dagger(A) \subseteq A$ full subanima implies the same morphism in $D(A)$ induces isomorphism on H^i with $i < 0$.[1][2] It follows from the LES of cohomology that This implies $A^{\dagger\mathrm{red}}$ is static. We thus have $A^{\dagger\mathrm{red}}$ as a quotient of $\pi_0 A$ in static modules. So we have ring structure by hand. ☐

Example : For A animated $R_\square$-algebra with $\pi_0 A$ separated π-adically, then $\pi_0 \mathrm{Nil}^\dagger(A) = 0$.

Proof. All maps $R_\square\langle\mathbb{N}[S]\rangle \to A$ must factor through some $R_\square\langle\mathbb{N}[S/\pi^n]\rangle \to A^\circ$. Then π^n divides the image $S \to \pi_0(A^\circ)$ for all n. So it's zero by π-adically separated. ☐

Example : $A = K\langle T_1, \dots, T_n\rangle/I$ for K NA field. Then A^{red} is π-adically separated. This implies $\mathrm{Nil}^\dagger(A) \subseteq \mathrm{Nil}(A)$, the usual algebraic nilradical. Of course, the other inclusion is true, and thus $\mathrm{Nil}^\dagger(A) = \mathrm{Nil}(A)$.

Example : A Tate-Huber ring, $I \subseteq A$ ideal, $\overline{I}$ topological closure. Assume $(A/\overline{I})^\circ$ is π-adically separated. So $\mathrm{Nil}^\dagger(A/I) = \overline{I}/I$.

Now for affinoid rings, recall that an analytic ring $\mathcal{A}$ is a pair $(A, D(\mathcal{A}))$ where A is an animated condensed ring and $D(\mathcal{A})$ is a full subcategory of $D(A)$ such that certain conditions. Point is, suffices to specify A and "free complete objects", a functor on ext.dis. to $D(A)$ with natural transformation

$$\begin{tikzcd}
A[S] \arrow[r] & \mathcal{A}[S] \\
S \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We will assume A is "complete", meaning $\mathcal{A}[*] \simeq A$.

Examples : A ring finite type then $A_\square[S] := \varprojlim_i A[S_i]$ where $S = \varprojlim_i S_i$ with S_i finite gives an analytic ring structure on A.

For general rings A, $A_\square[S] := \varinjlim_{A' \to A} A'_\square[S]$ where $A' \to A$ ranges over finite type rings mapping into A.

Example : A animated $\mathbb{Z}_\square$-algebra and $A^+ \subseteq \pi_0(A^\circ)(*)$ a subring, assume A complete in $(A, A^+)_\square$. We get an analytic ring structure on $(A, A^+)_\square$. Emanuel does not describes this in general but for static A, the definition $(A, A^+)_\square[S] := A[S] \otimes_{A^+} A^+_\square$ works. [3]

> **Definition**
> Let $\mathcal{A} = (A, D(\mathcal{A}))$ be an analytic ring over $\mathbb{Z}_\square$. Then
>
> 1. $\mathcal{A}^+ := \mathrm{Map}_{\mathrm{AniRing}_\square}(\mathbb{Z}[T]_\square, \mathcal{A}) \subseteq \mathrm{Map}(\mathbb{Z}[T]_\square, \mathcal{A}) = A(*)$.
> 2. Define $\mathcal{A}^{\circ\circ}, \mathcal{A}^\circ, \mathcal{A}^b$ by the corresponding thing for A.

---

[1] Cohomological convention.

[2] I guess an argument for this is that this is true iff forget to derived category of condensed abelian groups. Then use Dold–Kan (i.e. the abelian group structure) to see that $H^{i<0}$ is independent of base point in A, whether in $\mathrm{Nil}^\dagger(A)$ or not.

[3] This formula *should* work, but does not quite work on the nose, as a member of the audience points out.

Fact : $\mathcal{A}^+ \subseteq A(*)$ is a subanima so we get ring structure on $\mathcal{A}^+$.

Example : For (A, A^+) Tate-Huber pair we have an analytic ring $\mathcal{A} := (A, A^+)_\square$ with $\mathcal{A}^+ = A^+$. (Andreychev)

> **Definition**
> A solid affinoid ring is an analytic ring $\mathcal{A}/\mathbb{Z}_\square$ such that $(A, \pi_0\mathcal{A}^+)_\square \to \mathcal{A}$ is an equivalence.

Counterexample : ultrasolid rational numbers = complement of the idempotent $\mathbb{Z}_\square$-algebra $\widehat{Z} = \prod_p \mathbb{Z}_p$ is not solid affinoid. (See Clausen-Scholze.)

Example : $\mathbb{Z}[T]_\square$ with $\mathcal{A}^+ = \mathbb{Z}[T]$.

Example : $\mathbb{Z}_\square[\mathbb{N}[S]]$ with $\mathcal{A}^+ = \mathbb{Z}$.

> **Proposition**
> 1. Given $\mathcal{A}$ solid affinoid over $\mathbb{Z}_\square$, then $\mathcal{A}^{\circ\circ}$ is a solid $\mathcal{A}^+_\square$-module.
> 2. Given $\mathcal{A}$ solid affinoid over $R_\square$ [a] then
>   (a) $\mathcal{A}^\circ$ is a solid $\mathcal{A}^+_\square$-algebra
>   (b) $\mathcal{A}^b$ is a solid $\mathcal{A}^+_\square$-algebra
>   (c) $\mathrm{Nil}^\dagger(\mathcal{A})$ is a solid $\mathcal{A}^b$-module and hence $\mathrm{Nil}^\dagger(A) \subseteq A^b$ is a full subanima ideal.
>
> ---
> [a] I guess implicitly there's an analytic ring structure on $R_\square$ as well.

Proof. Apparently just diagram chases. ☐

> **Definition**
> Let $\mathcal{A}$ be a solid affinoid ring over $R_\square$. Then we say $\mathcal{A}$ is bounded when A is a bounded animated $R_\square$-algebra.
>
> Now assume $\mathcal{A}$ is bounded. Give $A^{\dagger\mathrm{red}}$ the analytic ring structure by declaring $M \in D(A^{\dagger\mathrm{red}})$ to be complete when they are $N \otimes^L_A A^{\dagger\mathrm{red}}$ for some $N \in D(\mathcal{A})$. This defines $\mathcal{A}^{\dagger\mathrm{red}}$.
>
> We declare $\mathcal{A}$ to be †-reduced when $\mathcal{A} \to \mathcal{A}^{\dagger\mathrm{red}}$ is an equivalence.
>
> We have fullsubcategories : reduced inside bounded inside affinoid.

Example : (A, A^+) Tate-Huber pair. Then $(A, A^+)_\square$ is a bounded affinoid ring over $R_\square$.

> **Proposition**
> $\mathrm{AffRing}^b_{R_\square}$ is stable under small colimits, finite products in the infinity category of analytic rings over $R_\square$.

Proof. Tensor products : Analytic rings over $R_\square$ have tensor products. To check the underlying animated condensed ring is bounded, use criterion (2) from before again. ☐

Proposition

We have the following :

1. Solid affinoid structures are independent of $\mathrm{Nil}^{\dagger}$, in the sense that for $\mathcal{A} \in \mathrm{AffRing}^{b}_{R_{\square}}$,

$$\begin{tikzcd}
R_{\square}[T] \arrow[rr] \arrow[dr] & & \mathcal{A} \arrow[rr] & & \mathcal{A}^{\dagger\mathrm{red}} \\
& & R\langle T\rangle_{\square} \arrow[u, dashed, "\exists\Leftrightarrow"'] \arrow[urr, dashed, "\exists"] & := & R_{\square} \otimes \mathbb{Z}[T]_{\square}
\end{tikzcd}$$

2. $\_^{\dagger\mathrm{red}}$ is idempotent.

4 Derived Tate spaces I (Anschütz)

R continues to be $(\mathbb{Z}((\pi)), \mathbb{Z}[[\pi]])_{\square}$. Aim : for $\mathcal{A} \in \mathrm{AffRing}^{b}_{R_{\square}}$, construct $\delta_{\mathcal{A}} : \mathrm{Spec}^{\mathrm{big}}\, D(\mathcal{A}) \to |\mathrm{Spa}\, \mathcal{A}|$ where the latter is a spectral space, defining a categorified locale $\mathrm{Spa}\, \mathcal{A} := (D(\mathcal{A}), |\mathrm{Spa}\, \mathcal{A}|, \delta_{\mathcal{A}})$. This should generalise the one for classical Tate-Huber pairs (A, A^{+}).

Recall, for classical Tate-Huber pairs,

$$\begin{aligned}
\mathrm{Spa}\,(A, A^{+}) &\to \mathrm{Spec}^{\mathrm{big}}(D((A, A^{+})_{\square})) \\
\{|f| > 1\} &\mapsto \mathbb{Z}((1/T)) \otimes_{\mathbb{Z}_{\square}[T], T\mapsto f} (A, A^{+})_{\square} \\
\{|f| < 1\} &\mapsto \mathbb{Z}[[T]] \otimes_{\mathbb{Z}_{\square}[T], T\mapsto f} (A, A^{+})_{\square}
\end{aligned}$$

these all land in bounded affinoid algebras over $R_{\square}$.

Example : For $\mathcal{A} := (R\langle T\rangle, R^{+}\langle T\rangle)_{\square}$, we have an idempotent algebra $R\langle T\rangle[1/T]$ which is *not* bounded. Geometrically, the corresponding open is the non-affine subspace given by formal completion at the origin.

Definition

Let $\mathcal{A} \in \mathrm{AffRing}^{b}_{R_{\square}}$. We define $|\mathrm{Spa}\ \mathcal{A}|$ as follows. For $I \subseteq \pi_0\mathcal{A}^{\circ}$ finite subset, it induces

$$(R\langle T_I\rangle, R)_{\square} \to \mathcal{A}, T_i \mapsto i$$

which induces

$$\mathrm{Spec}^{\mathrm{big}}\, D(\mathcal{A}) \to \mathrm{Spec}^{\mathrm{big}}(R\langle T_I\rangle, R)_{\square} \to \left|\mathrm{Spa}^{\,\mathrm{cl}}(R\langle T_I\rangle, R)\right|$$

where the final spectral space is the classical adic spectrum. We now take inverse limit along all finite subset $I \subseteq \pi_0 A^{\circ}$.

$$\rho_{\mathcal{A}} : \mathrm{Spec}^{\mathrm{big}}\, D(\mathcal{A}) \to \tau_{\mathcal{A}} := \varprojlim_{I \subseteq \pi_0 A^{\circ}} \left|\mathrm{Spa}^{\,\mathrm{cl}}(R\langle T_I\rangle, R)\right|$$

where the latter limit in locales. [a]

---

[a] It turns out in this situation, this is also the limit in spectral spaces.

> **Proposition**
> There exists a maximal open subspace $U \subseteq \tau_{\mathcal{A}}$ in the constructible topology such that $\rho_{\mathcal{A}}$ factors into a surjective map of locales.
> $$\delta_{\mathcal{A}} : \operatorname{Spec}^{\text{big}} D(\mathcal{A}) \to \tau_{\mathcal{A}} \setminus U := |\text{Spa } \mathcal{A}|$$

Question from audience : What is a surjection of locales? Anschütz : I will get to it.

We can work in a more general context : $\mathcal{C} \in \text{cAlg}(\text{Pr}^{L}_{st})$ with $\otimes$ preserving compact objects, with compact unit, and $\rho : \operatorname{Spec}^{\text{big}} \mathcal{C} \to \tau$ with τ spectral space. We further assume $Z \subseteq \tau$ closed and constructible, then $\mathcal{A}(Z) := \rho^{-1}Z \in \mathcal{C}$ is compact. Then there exists maximal subspace $U \subseteq \tau$ open in the constructible topology such that ρ factors as $\operatorname{Spec}^{\text{big}} \mathcal{C} \to \tau \setminus U$.

Anschütz goes on to construct U in this generality. Idea : $t \in U$ are the points which contribute nothing in terms of $\mathcal{C}$. Involves defining for $V \subseteq \tau$ constructible, a quotient $j_V^* : \mathcal{C} \to \mathcal{C}_V$ such thatj_V^* is

1. j_V^* is symmetric monoidal and $j_{*,V}$ commutes with colimits.
2. the natural quotient if V is open
3. $\mathcal{A}(V) \otimes \_$ if V is closed
4. $\mathcal{C}_{V \cap W} = \mathcal{C}_V \otimes_{\mathcal{C}} \mathcal{C}_W$.

Then one can define the "stalk category" for $t \in \tau$,

$$\mathcal{C}_t := \varinjlim_{t \in V \text{ constructible}} \mathcal{C}_V$$

Then $U := \{t \in \tau \text{ s.t. } \mathcal{C}_t = 0\}$. Anschütz then proves this this gives the desired result. The surjectivity is in the sense that for $Z, Z_1 \subseteq |\text{Spa}\,\mathcal{A}|$ closed subspaces, to check $Z = Z_1$ it suffices to check $\delta_{\mathcal{A}}^{-1} Z = \delta_{\mathcal{A}}^{-1} Z_1$.

Now we assume $\mathcal{C} := D(\mathcal{A})$ where $\mathcal{A} \in \text{AffRing}^{b}_{R_\square}$. One can check the conditions by reducing to the case of usual adic spaces.

Remark. For $x \in |\text{Spa } \mathcal{A}|$, we had $D(\mathcal{A})_x := \varinjlim_{x \in V} D(\mathcal{A})_V$ where V is constructible. It turns out $D(\mathcal{A})_V = D(\mathcal{A}_V)$ for some bounded affinoid ring $\mathcal{A}_V$. Furthermore,

$$\varinjlim_{x \in V} D(\mathcal{A})_V \simeq D(\varinjlim_{x \in V} \mathcal{A}_V)$$

where $\mathcal{A}(x)_{\text{cons}} := \varinjlim_{x \in V} \mathcal{A}_V$ is called the *constructible stalk*. The *adic stalk* is analogous, but only along rational neighbourhoods.

Take example of $x \in \operatorname{Spa}^{\text{cl}}(R\langle T\rangle, R^+) \setminus \operatorname{Spa}^{\text{cl}}(R\langle T\rangle, R^+\langle T\rangle)$. Then $\mathcal{A}(x)_{\text{cons}} = R\langle T\rangle \otimes_{\mathbb{Z}[T]} \mathbb{Z}((1/T))$. Concretely, this is the ring of $\sum_{n\in\mathbb{Z}} r_n T^n$ where

1. $r_n \to 0$ as $n \to -\infty$
2. r_n bounded as $n \to \infty$.

Question from audience : is there a similar description for the adic stalk in this example?

Anschütz : Not that I know of.

5 Six functor formalisms I (Lukas Mann)

Definition

A geoemtric setup consists of

1. (Spaces) $\mathcal{C}$ an ∞-category with finite limits
2. E a class of morphisms in $\mathcal{C}$ such that
    (a) stable under composition
    (b) stable under pullback. More precisely given $X \to S$ in E and $T \to S$ then $X \times_S T \to T$ is in E.
    (c) diagonals meaning for all $X \to S$ we have $X \to X \times_S X$ is in E. Equivalently under the previous assumptions, given $X \to Y \to S$ with $X \to S$ and $Y \to S$ in E then $X \to Y$ is also.

A morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{C}', E')$ consists of a functor $F : \mathcal{C} \to \mathcal{C}'$ such that

1. $F(E) \subseteq E'$
2. F preserves pullbacks in E

Definition

Let $(\mathcal{C}, E)$ be a geometric setup. Let $\mathrm{Corr}(\mathcal{C}, E)$ be the following symmetric monoidal ∞-category : [a]

- Objects are those of $\mathcal{C}$.
- $\mathrm{Corr}(\mathcal{C}, E)(X, Y)$ are correspondences $X \leftarrow Z \to Y$ where $Z \to Y$ is in E.
- Composition is given by fiber product of correspondences.
- for $X, Y \in \mathrm{Corr}(\mathcal{C}, E)$, define $X \otimes Y := X \times Y$.

---

[a] For the talk, we will only describe objects and morphisms. See, for example Scholze's notes, for precise definition as an ∞-category.

Remark – Variant not in Juan's paper. Suppose we have $I, P \subseteq E$ such that $(\mathcal{C}, I), (\mathcal{C}, P)$ also give geometric setups. Then there is a $(\infty, 2)$-category $\mathrm{Corr}^{P,I}(\mathcal{C}, E)$ where a 2-morphism is

$$\begin{tikzcd}
 & X & \\
Z \arrow[ur, "\in E"] \arrow[dr] & W \arrow[l, "\in I"'] \arrow[r, "\in P"'] & Z_1 \arrow[ul, "\in E"'] \arrow[dl] \\
 & Y &
\end{tikzcd}$$

Definition

> A 3-functor formalism on a geometric setup $(\mathcal{C}, E)$ is a lax symmetric monoidal functor
>
> $$D : (\mathrm{Corr}(\mathcal{C}, E), \otimes) \to (\infty\mathrm{Cat}, \times)$$

From such D we can derive the following data :

- ("sheaves") for each $X \in \mathcal{C}$ an infinity category $D(X)$
- Given $f : X \to Y$, $f^* : D(Y) \to D(X)$ given by image of $Y \leftarrow X = X$.
- Given $f : X \to Y$ in E then $f_! : D(X) \to D(Y)$ given by image of $X = X \to Y$.
- D lax monoidal and $X \in \mathcal{C}$ commutative algebra induces a commutative algebra structure on $D(X)$ in $(\infty\mathrm{Cat}, \times)$. Furthermore f^* is symmetric monoidal for all f.
- Base change for E.
- Projection formula for E.

Remark. This is not agreed upon in the literature, however there should be a notion of lax symmetric monoidal functor of $(\infty, 2)$-categories. Then a 3-functor formalism for $(\mathcal{C}, E, P, I)$ will be a lax symmetric monoidal functor

$$D : \mathrm{Corr}^{P,I}(\mathcal{C}, E) \to \infty\mathrm{Cat}$$

such that

1. For $j \in I$ then $j_! \dashv j^*$.
2. For $f \in P$ then $f^* \dashv f_!$.

with fixed units and counits for the above adjunctions.

> **Definition**
>
> A 6-functor formalism is a 3-functor formalism D on $(\mathcal{C}, E)$ such that the right adjoints of $f^*, f_!$ exist and each $D(X)$ is closed symmetric monoidal.

> **Proposition – In progress**
>
> Let $(\mathcal{C}, E)$ be a geometric setup, $I, P \subseteq E$ a *suitable decomposition*, meaning
>
> 1. $(\mathcal{C}, I), (\mathcal{C}, P)$ are geometric setups
> 2. All $f \in I \cap P$ then f is truncated. [a]
> 3. Every $f \in E$ can be written as $f = pi$ where $p \in P$ and $i \in I$.
>
> Then the restriction on
>
> $$\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathrm{Corr}^{P,I}(\mathcal{C}, E), \infty\mathrm{Cat}) \to \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\mathrm{op}}, \infty\mathrm{Cat}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{cAlg}(\infty\mathrm{Cat}))$$
>
> is fully faithful (on the underlying ∞-category?) and the essential image consists of $D : \mathcal{C}^{\mathrm{op}} \to \mathrm{cAlg}(\infty\mathrm{Cat})$ such that
>
> 1. For $j \in I$, j^* has left adjoint $j_!$ with base change and projection formula.

2. For $p \in P$, p^* has right adjoint $p_!$ with base change and projection formula.
3. $p_!$ and j^* in the above are compatible in the sense that for all

$$\begin{tikzcd} Y \arrow[r, "j'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ T \arrow[r, "j"'] & S \end{tikzcd}$$

then the induced morphism $f'_! j_! \to j'_! f_!$ is an equivalence.

---

[a] From audience : This is fine for $(n, 1)$-categories for all $n < \infty$.

Definition

Let D be a 3-functor formalism on $(\mathcal{C}, E)$.

1. A family $(U_i \to X)_i$ is a universal $*$-cover when $D^* : \mathcal{C}^{\mathrm{op}} \to \infty\mathrm{Cat}$ descends universally along it. [a]
2. A family $(U_i \to X)_i$ in E is a universal !-cover when $D^! : \mathcal{C}_E^{\mathrm{op}} \to \infty\mathrm{Cat}$ descends along it. [b]

---

[a] $D(X)$ is the totalisation of D applied to the Cech nerve of $(U_i \to X)_i$ and also for any base change of the cover along any $Y \to X$.

[b] Mann cheated a bit here. See Scholze's notes, definition 4.14 for the fine print.

Proposition

Let D be a 6-functor formalism on $(\mathcal{C}, E)$ with values in Pr^L. Assume $\mathcal{C}$ is equipped with a subcanonical site structure τ and D^* descends along τ-covers. Then there exists a geometric setup $(\mathcal{X}, E')$ satisfying the following :

1. $\mathcal{X} := \mathrm{Sh}(\mathcal{C}, \tau)$
2. E' is $*$-local on target : For $f : Y \to X$ in $\mathcal{X}$ in E' after pullback along every object in $\mathcal{C}$ then f is in E'.
3. E' is !-local on source and target. (If f is in E' after passing to universal !-covers of the source or target then $f \in E'$)

Example : $\mathcal{C} = \mathrm{Sh}(\mathrm{ProFin}, \infty\mathrm{Grpd}) = \mathrm{CondAni}$. $D = D(\_, \Lambda)$ Λ-valued sheaves. E' is "Λ-fine" maps. This includes

1. maps between manifolds
2. maps $*/G \to */H$ for qcqs G, H, or p-adic Lie groups.

Example : $\mathcal{C}$ = stacks on schemes, $D(X) := D_{\text{ét}}(X, \Lambda)$ where Λ is a torsion ring.

Example : $\mathcal{C}$ = analytic stacks, D is quasi-coherent sheaves.

6 Algebraic topology from the six functor point of view (Lars Hesselholt)

(Writing $\mathcal{C}$ too tiring so I switched to C. Also, I was unable to follow this lecture well so there may be more mistakes in this section, I apologise.)

We take $D : \mathrm{Corr}(C, E) \to \mathrm{Pr}^L$ lax symmetric monoidal. Then we have lax symmetric monoidal :

$$C^{\mathrm{op}} \longrightarrow \mathrm{Corr}(C, E) \longrightarrow \mathrm{Pr}^L$$

which induces

$$\mathrm{cAlg}(C^{\mathrm{op}}) \to \mathrm{cAlg}(\mathrm{Corr}(C, E)) \to \mathrm{cAlg}(\mathrm{Pr}^L)$$

Let $S \in C$ be final. Then given $f : X \to S$ and $V \in D(S)$,

1. $f_* f^*(V)$ is called cohomology of X with coefficients in V
2. $f_! f^*(V)$ is called cohomology of X with compact support and coefficients in V
3. $f_* f^!(V)$ is Borel–Moore homology of X with coefficients in V
4. $f_! f^!(V)$ is homology of X with coefficients in V.

Cohomology is functorial in spaces. At least, given

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, "q"'] & X \arrow[d, "p"] \\
 & S
\end{tikzcd}$$

then we have

$$p_* p^*(V) \to p_* f_* f^* p^*(V) \simeq q_* q^*(V)$$

How can we make this functorial?

> **Proposition**
>
> Let $C \in \widehat{\infty\mathrm{Cat}}$.[a]
>
> 1. The assignment
>
> $$(C \xrightarrow{f} D) \mapsto (C \xrightarrow{f_* f^*} C)$$
>
>   promotes to a functor
>
> $$\left(\widehat{\infty\mathrm{Cat}}^{L}\right)_{C/} \to \mathrm{Fun}(C, C)$$
>
> 2. The assignment
>
> $$(D \xrightarrow{f_!} C) \mapsto (C \xrightarrow{f_! f^!} C)$$

> promotes to a functor
>
> $$\left(\widehat{\infty\mathrm{Cat}}^{L}\right)_{/C} \to \mathrm{Fun}(C, C)$$
>
> ---
>
> [a] Large ∞-categories.

Remark. [Lur08, Remark 7.1.6.6] says the above is possible, but does not provide a proof.

For $C := \mathrm{Ani}$ with $X \in \mathrm{Ani}$, $D(X) := \mathrm{Fun}(X, \mathrm{Sp})$. Then $f_!, f_*, f^*, f^!$ can be described as Kan extensions.

$$\begin{tikzcd}
f : X \to Y & \rightsquigarrow & \mathrm{Fun}(X, \mathrm{Sp}) \arrow[r, shift left=6, "f_!"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, leftarrow, "f^! \simeq f^*" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "f^*"'] & \mathrm{Fun}(Y, \mathrm{Sp})
\end{tikzcd}$$

Note to self : This is completely analogous to the triple of adjoints where X, Y are categories and one has Set instead of Sp.

$$\mathrm{Corr}(\mathrm{Ani}) \xrightarrow{D} \mathrm{Pr}^{L}_{st}$$

1. I all maps
2. P maps with fibers which are compact projective anima, A.K.A. finite sets

> **Proposition – Generalised Mayer–Vietoris**
> The functor
>
> $$\begin{tikzcd}
\mathrm{Ani}^{\mathrm{op}} \arrow[r] & \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp}) \\
(p : X \to 1) \arrow[r, mapsto] & p_* p^*
\end{tikzcd}$$
>
> takes colimits in Ani to limits in $\mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$.

Proof. Let $X : K^{\triangleright} \to \mathrm{Ani}$ be a colimit diagram and let $1 : K^{\triangleright} \to \mathrm{Ani}$ be the terminal diagram. Let $p : X \to 1$ be the unique map. We want to show

$$(K^{\mathrm{op}})^{\triangleleft} \simeq (K^{\triangleright})^{\mathrm{op}} \xrightarrow{p_* p^*} \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$$

is a limit diagram. Equivalently, for all $V \in \mathrm{Sp}$,

$$(K^{\mathrm{op}})^{\triangleleft} \xrightarrow{p_* p^*(V)} \mathrm{Sp}$$

is a limit diagram. By Yoneda, it suffices to show that for all $W \in \mathrm{Sp}$, the diagram of anima

$$(K^{\mathrm{op}})^{\triangleleft} \xrightarrow{\mathrm{Map}(W, p_* p^* V) \simeq \mathrm{Map}(p^* W, p^* V)} \mathrm{Ani}$$

is a limit diagram. But $D^* : \mathrm{Ani}^{\mathrm{op}} \to \mathrm{Pr}^L_{st}$ preserves limits and mapping anima in a limit of ∞-categories is the limit of the mapping anima, since

$$\begin{tikzcd}
\mathrm{Map}(Y, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\
1 \arrow[r, "{(Y,X)}"'] & C \times C
\end{tikzcd}$$

□

Proposition – Universal coefficient theorem
For $V \in D(S)$ we have

$$\mathrm{Map}(f_! f^!(1), V) \simeq f_* f^* V$$

Proof.

$$\begin{aligned}
\mathrm{Map}(f_! f^! 1, V) &\simeq \mathrm{Map}(f^! 1, f^! V) \\
&\simeq \mathrm{Map}(f^! 1 \otimes f^* 1, f^1 1 \otimes f^* V) \\
&\simeq \mathrm{Map}(f^* 1, f^* V) \\
&\simeq f_* f^* V
\end{aligned}$$

where we used f smooth implies $f^!(\_) \simeq f^! 1 \otimes f^*(\_)$. The object $f^! 1$ is invertible. □

There are two dualities in algebraic topology. Smooth duality concerns “conorm” map

$$f^! 1 \otimes f^*(\_) \to f^!(\_)$$

defined to be the mate / adjunct of

$$f_!(f^! 1 \otimes f^*(\_)) \simeq f_! f^! 1 \otimes \_ \xrightarrow{\epsilon} \_$$

Proper duality conerns the norm map

$$f_!(\_ \otimes p_{2*} \Delta_! 1) \xrightarrow{\mathrm{Nm}_f} f_*(\_)$$

defined as the mate / adjunct of

$$f^* f_!(\_ \otimes p_{2*} \Delta_1(1)) \simeq p_{1!} p_2^*(\_ \otimes p_{2*} \Delta_!(1)) \simeq p_{1!}\left(p_2^*(\_) \otimes p_2^* p_{2*} \Delta_1(1)\right) \xrightarrow{\varepsilon} p_{1!}\left(p_2^*(\_) \otimes \Delta_1(1)\right) \simeq p_{1!} \Delta_! \Delta^* p_2^*(\_) \simeq \_$$

where

$$\begin{tikzcd}
Y \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dashed, "\Delta"] & & \\
& Y \times_X Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd}$$

The object $p_{2*}\Delta_!(1)$ is called the Klein–Spivalz dualizing object. Poincaré duality in algebraic topology is proper duality, not smooth duality.

Lu–Zheng magic :

$$\begin{tikzcd}
\mathrm{Corr}(C) \arrow[rr, "D"] \arrow[rd, "D'"'] & & \mathrm{Mod}_{D(S)}\mathrm{Pr}^L \\
& K_{D,S} \arrow[ru, "D''"'] &
\end{tikzcd}$$

In $\mathrm{Corr}(C)$ the mapping object is $\underline{\mathrm{Map}}(X, Y) \simeq X \times_S Y$.

1. $K_{D,S}$ has same objects as $\mathrm{Corr}\, C$ but for hom Y to X is $D(X \times_S Y)$.
2. D'' does the same as D on objects but on homs gives

$$\begin{aligned} D(X \times_S Y) &\to \mathrm{Fun}^L_{D(S)}(D(Y), D(X)) \\ K &\mapsto p_{1!}(p_2^*(\_) \otimes K) \end{aligned}$$

> **Definition – Mann**
> Let $f : X \to S$ and $P \in D(X)$.
>
> 1. P is f-smooth when P as a morphism $X \to S$ in $K_{D,S}$ is a left adjoint.
> 2. P is f-proper when P as a morphism $X \to S$ in $K_{D,S}$ is a right adjoint.

The above uses that D'' preserves adjunctions, which itself comes from D'' secretly being a functor of $(\infty, 2)$-categories.

What does this look like? For f-smooth P,

$$\begin{tikzcd}
X \arrow[r, shift left, "P"] & S \arrow[l, shift left, "Q"] & \rightsquigarrow & D(X) \arrow[r, shift left, "f_!(\_\otimes P)"] & D(S) \arrow[l, shift left, "f^*(\_)\otimes Q"]
\end{tikzcd}$$

$f_!(\_ \otimes P)$ has $\underline{\mathrm{Map}}(P, f^!\_)$ as right adjoint. So if P has Q as right adjoint then

$$\underline{\mathrm{Map}}(P, f^!(\_)) \simeq f^*(\_) \otimes Q$$

For f-proper P,

$$\begin{tikzcd} X \arrow[r, shift left, "Q"] & S \arrow[l, shift left, "P"] & \rightsquigarrow & D(X) \arrow[r, shift right, "f_!(\_\otimes P)"'] & D(S) \arrow[l, shift right, "f^*(\_)\otimes Q"'] \end{tikzcd}$$

$f^*(\_) \otimes Q$ has $f_*(\underline{\mathrm{Map}}(Q, \_))$ as right adjoint. So if Q is left adjoint of P then

$$f_*(\underline{\mathrm{Map}}(Q, \_)) \simeq f_!(\_ \otimes P)$$

A useful result (Heyer–Mann) : Pointwise criterion for adjoints inside a $(\infty, 2)$-category.

The above implies the property of $f : X \to S$ that the norm map

$$f_!(\_ \otimes p_{2*}\Delta_!(1)) \xrightarrow{\mathrm{Nm}_f} f_*(\_)$$

being an equivalence is stable under base change along any $g : S' \to S$.

Exercise : Show that if $f : Y \to X$ a morphism of anima has compact fibers, then Nm_f is an equivalence. (Hint : Begin with finite fibers.)

7 Derived Tate spaces II (Anschütz)

Proposition

Let $A \to B$ in $\mathrm{AffRing}^b_{R_\square}$.

1. $|\mathrm{Spa}\ A|$ is a spectral space, with topological basis given by rational subsets. (Rational subset = pullback of rational subset along $|\mathrm{Spa}\ A| \to \left|\mathrm{Spa}^{\mathrm{cl}}(R\langle T_I\rangle, R^+)\right|$ where $I \subseteq \pi_0 A^\circ$ finite subset.)
2. $|\mathrm{Spa}\ B| \to |\mathrm{Spa}\ A|$ spectral map. In fact, pullback of rational subsets are rational subsets.[a][b]
3. If $A = (A, A^+)_\square$ with (A, A^+) a classical Tate–Huber ring, then $|\mathrm{Spa}\,(A, A^+)_\square| \simeq \left|\mathrm{Spa}^{\mathrm{cl}}(A, A^+)\right|$.

---

[a] Clear from construction.

[b] In classical adic spaces, the underlying topological morphism is generalising. However, this is *false* in the generality for bounded affinoid rings.

Proof. Uninteresting. ☐

Definition

Let $A \in \mathrm{AffRing}^b_{R_\square}$ and $x \in |\mathrm{Spa}\,A|$.

1. The residue field of A at x is defined as

$$\kappa(x) := A(x)^{\dagger\mathrm{red}}$$

> 2. The *constructible* residue field of A at x is
>
> $$\kappa(x)_{\mathrm{cons}} := A(x)^{\dagger\mathrm{red}}_{\mathrm{cons}}$$

> **Proposition**
> Let $A \in \mathrm{AffRing}^b_{R_\square}$. TFAE :
>
> 1. the open subsets of $|\mathrm{Spa}\ A|$ form a totally ordered set and non-empty.
> 2. $|\mathrm{Spa}\ A|$ has a unique closed point x
> 3. $|\mathrm{Spa}\ A| \neq \varnothing$ and for $f, g \in \pi_0 A \setminus \pi_0 \mathrm{Nil}^\dagger(A)$, then either $\{|f| \leq |g| \neq 0\}$ or $\{|g| \leq |f| \neq 0\}$ is equal to all of $|\mathrm{Spa}\ A|$.

Adic stalks of bounded affinoid rings satisfies these conditions and the residue and constructible residue fields are indeed fields.

Proof. $(1 \Rightarrow 2)$ ok. $(2 \Rightarrow 3)$ $|\mathrm{Spa}\ A|$ is the only open neighbourhood of x.

Claim : $\pi_0 A$ is local with maximal ideal $\pi_0 \mathrm{Nil}^\dagger(A)$. Let $f \in \pi_0 A \setminus \pi_0 \mathrm{Nil}^\dagger(A)$. If $x \in \{|f| \leq |\pi|^n\}$ for all n then $|\mathrm{Spa}\ A| = \{|f| \leq |\pi|^n\}$ for all n. This implies $R[T] \to A, T \mapsto f$ extends to $R\{T\}^\dagger \to A$ i.e. $f \in \mathrm{Nil}^\dagger(A)$. The argument for classical Tate–Huber pairs is intuitive. This lifts to a proof for our case by construction of $|\mathrm{Spa}\ A|$. This implies there exists n such that $x \notin \{|f| \leq |\pi|^n\}$. Then $|\mathrm{Spa}\ A| = \{|f| \geq |\pi|^n\}$, which implies $f \in (\pi_0 A)^\times$.

Now take f, g as in (3). Then consider $\{|f/g| \leq 1\} \cup \{|g/f| \leq 1\}$. Then x is in one of these. Then $|\mathrm{Spa}\ A|$ is whichever it is.

$(3 \Rightarrow 1)$ Skipped. ☐

> **Proposition**
> Let $A \in \mathrm{AffRing}^b_{R_\square}$. TFAE :
>
> 1. $|\mathrm{Spa}\ A|$ is a point
> 2. $|\mathrm{Spa}\ A|$ has a unique closed point x and $f \in \pi_0 A$, either $f \in A^+$ or ($f \in A^\times$ and $1/f \in A^{\circ\circ}$).

Constructible stalks of bounded affinoid rings satisfies these properties.

Proof. Uninteresting. ☐

> **Proposition**
> Let $(A_i)_i$ be a sifted diagram in $\mathrm{AffRing}^b_{R_\square}$ with colimit A. Then
>
> $$|\mathrm{Spa}\ A| \xrightarrow[\sim]{\alpha} \varprojlim_i |\mathrm{Spa}\ A_i|$$

> in topological spaces.

Proof. Key observation : $A(*) = \varinjlim_i A_i(*)$ because the point is a compact projective in condensed sets. By the construction of $|\mathrm{Spa}\ A|$, any constructible subset is the pullback of some constructible subset in some $|\mathrm{Spa}\ A_i|$. This implies α injective. Surjectivity follows formally from key observation applied to constructible stalk. $\square$

> **Proposition**
> Let $A \to B$ in $\mathrm{AffRing}^b_{R_\square}$, such that $B = B_{A/}$ meaning it has the induced analytic ring structure from A. Also assume $\pi_0 A \to \pi_0 B$ is surjective. Then $|\mathrm{Spa}\, B| \to |\mathrm{Spa}\, A|$ is an immersion. In fact, with $I := \mathrm{Fib}(A \to B)$ is generated by $\pi_0 I(*)$ and the image of $|\mathrm{Spa}\, B| = \{|f| = 0 \text{ s.t. } f \in I\}$.

The above applied in particular to $A \to A^{\dagger\mathrm{red}}$ so that $|\mathrm{Spa}\, A| = \left|\mathrm{Spa}\, A^{\dagger\mathrm{red}}\right|$.

> **Definition**
> Define $\mathrm{Aff}^b_{R_\square} := (\mathrm{AffRing}^b_{R_\square})^{\mathrm{op}}$. For $A \in \mathrm{AffRing}^b_{\square}$, we write $\mathrm{AnSpec}A$ for the corresponding opposite.
>
> A family $(\mathrm{AnSpec}A_i \to \mathrm{AnSpec}A)_i$ is an analytic cover when
>
> 1. each $\mathrm{AnSpec}A_i \to \mathrm{AnSpec}A$ is a categorical open immersion.
> 2. $\mathrm{AnSpec}A_i \to \mathrm{AnSpec}A$ are pulled back from rational subsets $U_i \subseteq |\mathrm{Spa}\, A|$ such that $\bigcup_i U_i = |\mathrm{Spa}\, A|$.
>
> This defines a Grothendieck site on $\mathrm{Aff}^b_{R_\square}$, which is in fact subcanonical.
>
> A derived Tate space is a sheaf X over $\mathrm{Aff}^b_{R_\square}$ in the analytic topology such that there exists a family of representable open subsheaves which maps jointly effective-epimorphically onto X.

Question to Anschütz : Is $(\mathrm{AnSpec}A_i \to \mathrm{AnSpec}A)_i$ covering on adic spectrum equivalent to $D(A) \to \prod_i D(A_i)$ descendable?

Anschütz : No I don't expect it to be equivalent. But we will see in later talks that it will satisfy $*$-descent.

There was a remark by Anschütz about how the above, by allowing categorical open immersions rather than just rational opens, we are actually allowing more ways to glue than Tate adic spaces in Huber's original theory.

8 Abstract 6-functor formalisms II (Lucas Mann)

In this talk, always assume E consists of all morphisms.

> **Definition**
> Let D be a 3-functor formalism on C. For $S \in C$ we let $K_{D,S}$ be the $(\infty, 2)$-category obtained by transferring the self-enrichment of $\mathrm{Corr}C$ along D. Explicitly :
>
> 1. objects = $C_{/S}$

> 2. $\mathrm{Hom}(Y, X) = D(X \times_S Y)$

There are natural $(\infty, 2)$-functors

$$C_{/S} \to \mathrm{Corr}(C_{/S}) \to K_{D,S} \xrightarrow{\mathrm{Hom}(S,\_)} \infty\mathrm{Cat}$$

> **Definition**
> A morphism $f : Y \to X$ in an $(\infty, 2)$-category K is a *left adjoint* when there exists $g : X \to Y$ and 2-morphisms $\eta : \mathbb{1}_Y \to gf$ and $\varepsilon : fg \to \mathbb{1}_X$ such that there exists commuting triangles[a]
>
> $$\begin{tikzcd} f \arrow[r, "f\eta"] \arrow[dr, "\mathrm{id}_f"'] & fgf \arrow[d, "\epsilon f"] & g \arrow[d, "\eta g"'] \arrow[dr, "\mathrm{id}_g"] & \\ & f & gfg \arrow[r, "g\epsilon"'] & g \end{tikzcd}$$
>
> ---
>
> [a] Technically, the commutation is data. But we are only asking for existence. It turns out that it is more or less unique, according to Lucas Mann.

Example : Let C be a symmetric monoidal ∞-category. $K := BC$ the $(\infty, 2)$-category with one object $*$ and $\mathrm{Hom}(*, *) = C$. Then TFAE for $P \in C$:

1. P is a left adjoint in K
2. P is left dualizable in C
3. $P \otimes \underline{\mathrm{Hom}}(P, 1) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, P)$.

> **Definition**
> Let D be a 3-functor formalism on C, $f : X \to S$ and $P \in D(X)$.
>
> 1. P is f-suave when it is a left adjoint as a map $X \to S$ in $K_{D,S}$. (Note $\mathrm{Hom}_{K_{D,S}}(X, S) \simeq D(X)$.)
> 2. f is D-suave when $1 \in D(X)$ is f-suave.
> 3. P is f-prim when it is a right adjoint as a map $X \to S$ in $K_{D,S}$.
> 4. f is D-prim when $1 \in D(X)$ is f-prim.

Note to self : suave = smooth, prim = proper. [1]

> **Proposition**
> Let D be a 3-functor formalism on C.
>
> 1. f-suave and f-prim are $*$-local on the target. If D is compatible with colimits ($D(X)$ has small colimits for all X and f^*, $f_!$ preserves them), then being suave is suave local on the source.

---

[1] Mann called these relatively dualisable and proper. Then Scholze replaced with smooth and proper. But étale and proper are dual. So Camargo called them smooth and co-smooth. But suave is like smooth except dualizing complex is not invertible. So people decided to just use completely different words. Suave is due to Scholze and prim is due to Hansen.

2. Passing to adjoint morphisms defines dualities SD_f and PD_f

$$\begin{aligned}\{\text{suave objects}\}^{\mathrm{op}} &\xrightarrow{\sim} \{\text{suave objects}\}\\ P &\mapsto \underline{\mathrm{Hom}}(P, f^!1)\end{aligned}$$

$$\begin{aligned}\{\text{prim objects}\}^{\mathrm{op}} &\xrightarrow{\sim} \{\text{prim objects}\}\\ P &\mapsto p_{2*}\underline{\mathrm{Hom}}(p_1^*P, \Delta_{f!}1)\end{aligned}$$

3. Suave $/S$ is stable under suave pullback and prim !-pushforward. Explicitly, given

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

   Then g suave implies $g^* : D(X) \to D(Y)$ sends f-suave to h-suave. Also g prim implies $g_! : D(Y) \to D(X)$ sends h-suave to f-suave. Similarly, prim $/S$ is stable under prim pullback and suave !-pushforward.

4. If f suave then

$$f^* \otimes \omega_f \xrightarrow{\sim} f^!$$

$$f^* \xrightarrow{\sim} \underline{\mathrm{Hom}}(\omega_f, f^!)$$

   where $\omega_f := \mathrm{SD}_f(1) = f^!1$. If f is prim then

$$f_! \xrightarrow{\sim} f_*\underline{\mathrm{Hom}}(\delta_f, \_)$$

$$f_!(\delta_f \otimes \_) \xrightarrow{\sim} f_*$$

   where $\delta_f := \mathrm{PD}_f(1) = p_{2*}\Delta_!1$. (Codualising complex.)

5. If f is suave then dualizable in $D(X)$ implies f-suave. If Δ_f suave then the converse is true.

   Same for prim.

6. Suppose $1 \in D(S)$ is compact. Then f-prim implies compact.

7. If $P \in D(X)$ is f-suave and $Q \in D(X)$ is f-prim then

$$f_*\underline{\mathrm{Hom}}(Q, \mathrm{SD}_f P) = (f_*\underline{\mathrm{Hom}}(\mathrm{PD}_f Q, P))^\vee$$

   where $\_^\vee = \underline{\mathrm{Hom}}(\_, 1)$ in $D(S)$.

Proof. Uninteresting. All formal 2-categorical arguments. ☐

> **Proposition – How to detect !-covers** 1. Every suave $*$-cover is a universal !-cover.
>
> 2. Let $g : Z \to Y$ be *descendable covers*, i.e. g is prim and $g_* 1 \in D(Y)$ is descendable. Then g is a universal !-cover.
>
>   Moreover for $f : Y \to X$, then $P \in D(Y)$ is f-prim iff $g^* P$ is fg-prim.

> **Definition**
>
> f is D-smooth when f is suave and ω_f is invertible. [a]
>
> f is D-cosmooth when f is prim and δ_f is invertible.
>
> f is D-étale whern it's truncated and f with its iterated diagonals are suave. (This turns out to imply $\omega_f \simeq 1$ and $f^! = f^*$.)
>
> Similar for D-proper. (Implies $\delta_f = 1$ and $f_! = f_*$.)
>
> ---
>
> [a] Under f suave, ω_f invertible iff ω_f dualizable.

Example : In $\mathrm{CondAni}$, $D = D(\_, \Lambda)$, then all Λ-fine maps between compact Hausdorff spaces are proper, topological manifolds are smooth (checking this reduces to $\mathbb{R}$). Open immersion are étale.

If X is locally compact Hausdorff, then $P \in D(X, \Lambda)$ is prim for $X \to *$ iff P is compact iff P is locally constant with perfect[1] stalks and compact support.

Mann : Also, constructible $P \in D(X)$ are suave over $*$, but there are conditions on the stratification. Maybe stratification by manifolds.

If G is a p-adic Lie group, then $V \in D(*/G, \Lambda)$ is suave over $*$ iff V admissible (i.e. V^K perfect for small enough compact open $K \subseteq G$). Similarly, V is prim over $*$ iff V is compact.

Example (the original motivating) : For $D_{\text{ét}}$, f-suave iff f-ULA. In [FS24], only a related honest 2-category is used.

Example : For $f : X \to S$ morphism of schemes with S finite type, then f-suave objects in $D_\square(X)$ is precisely $\mathrm{Coh}_S(X)$. "finite tor dimension over S".

If S regular Noetherian then $\mathrm{Coh}_S(X) = \mathrm{Coh}(X) =$ bounded complexes with finitely generated cohomologies. This allows calculation for suave morphisms in schemes.

9 Tate stacks (Guido Bosco)

$\mathrm{Aff}_{\mathbb{Z}_\square} := (\mathrm{AffRing}_{\mathbb{Z}_\square})^{\mathrm{op}}$.

Goal : "solid analytic stacks". But for what Grothendieck topology?

1. Want $A \mapsto D(A)$ to desend to a category of "quasi-coherent sheaves" $D(X)$ for each $X \in \mathrm{AnStack}_\square$.
2. Want 6-functor formalism for such quasi-coherent sheaves.

---

[1] If $\Lambda = \mathbb{Z}$, perfect means finitely generated.

First define 6-functor formalism on $C := \mathrm{Aff}_{\mathbb{Z}_\square}$.

Note to self : Technically, one needs work with a uncountable cut off cardinal κ and do all of the following for solid affinoids bounded by κ, checking at every step that κ-smallness is preserved and any functor the same up to enlarging κ.

> **Definition**
>
> A map $f : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is *proper* when $f_* : D(B) \to D(A)$ satisfies projection formula. Let P be the class of proper morphisms in C.
>
> A map $j : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is *open immersion* when j^* admits fully faithful left adjoint $j_!$ satisfying the projection formula. Use I to denote the class of open immersions in C.

> **Proposition**
>
> For $f : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$, f is proper iff $B = \underline{B}_{A/}$ the analytic ring structure induced from A.

Proof. The projection formula says : For all $M \in D(A)$ and $N \in D(B)$, $f_*(f^*M \otimes_B N) = M \otimes_A f_*N$ in $D(A)$.

If $B = B_{A/}$ then $f^* = \_ \otimes_A B$ and the projection formula follows from $(M \otimes_A B) \otimes_B N = M \otimes_A (N|_A)$.

Conversely, if we have the projection formula then applying to $N = \underline{\mathrm{B}}$ the underlying A-module of B gives $M \otimes_A B = M \otimes_A \underline{B}$ in $D(\underline{\mathrm{B}})$. This implies $B = \underline{B}_{A/}$.

□

Remark. This agree's with Huber's definition of partially proper upon fully faithfully embedding the 1-category of Tate-Huber pairs into $\mathrm{AffRing}_{\mathbb{Z}_\square}$.

> **Proposition**
>
> For $j : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$, j is an open immersion iff $j^* : D(A) \to D(B)$ is a categorical open immersion with corresponding idempotent algebra $D \in D(A)$ such that $\underline{\mathrm{Hom}}(D, \_)[1]$ commutes with small colimits and preserves (too small)
>
> Consequently, I is stable under composition and pullbacks. Furthermore, for $j \in I$, formation of $j_!$ commutes with base change.

Remark. Generally, $D = \mathrm{Cofib}(j_!1 \to 1)$. Conversely, given D we have $j_!(\_) = \mathrm{Fib}(A \to D) \otimes \_$.

Examples : $\mathrm{AnSpec}\mathbb{Z}[T]_\square \to \mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z})_\square$ and $\mathrm{AnSpec}(\mathbb{Z}[T^{\pm 1}], \mathbb{Z}[T^{\pm 1}])_\square \to \mathrm{AnSpec}\mathbb{Z}[T]_\square$. It's important to note that in analytic geometry, open immersions are really more like closed immersions in the classical sense.

> **Definition**
>
> For f a morphism in $\mathrm{Aff}_{\mathbb{Z}_\square}$, f is called !-able when $f = pj$ where $j \in I$ and $p \in P$.
>
> Denote the class of !-able morphisms by E.

Example : We have a factorisation

$$\mathrm{AnSpec}\mathbb{Z}[T]_\square \to \mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z})_\square \to \mathrm{AnSpec}\mathbb{Z}_\square$$

is a composition of open immersion followed by proper.

Very roughly, !-able corresponds to Huber's notion of "+ weakly of finite type".

> **Proposition**
>
> There is an enhancement of the functor
>
> $$D : C \to \mathrm{cAlg}(\mathrm{Pr}^L)$$
>
> to a symmetric monoidal 6 functor formalism
>
> $$D : (\mathrm{Corr}(C, E), \otimes) \to (\mathrm{Pr}^L, \otimes)$$

Proof sketch. One shows that (I, P) is a suitable decomposition for the geometric setup (C, E). Then one can use the criterion in Lucas' talk. Checking the conditions is uninteresting. □

Recall that for a !-able morphism $f : Y \to X$ in $\mathrm{Aff}_{\mathbb{Z}_\square}$, we say it satisfies !-descent when $D(X)$ is the limit of the image of the Cech nerve of f under $D^!$.

> **Proposition – !-descent topology**
>
> For $\mathrm{Aff}_{\mathbb{Z}_\square}$, define !-coverings as finite families $(Y_i \to X)_i$ of !-able morphisms with $\coprod_i Y_i \to X$ satisfying !-descent.
>
> Then $\mathrm{Aff}_{\mathbb{Z}_\square}$ with the !-coverings defines a Grothendieck topology.

> **Proposition**
>
> 1. If a !-able morphism in C satisfies !-descent, then it satisfies universal *-descent and universal !-descent. [a]
> 2. The !-descent topology on C is subcanonical.
>
> ---
>
> [a] For the proof, it is crucial that D is symmetric monoidal.

> **Proposition**
>
> Let D be a 6-functor formalism with values in Pr^L on a geometric setup (C, E). Let C be equipped with a subcanonical Grothendieck topology τ along which D^* descends. Then there is a geometric setup $(\mathrm{Sh}_\tau(C, \mathrm{Ani}), E')$ such that
>
> 1. $E \subseteq E'$
> 2. D extends uniquely to this geometric setup
> 3. E' is $*$-local on target
> 4. E' is !-local on target and source.

We apply this to $C = \mathrm{Aff}_{\mathbb{Z}_\square}$, E !-able, and τ the !-descent topology.

> **Definition**
> $\mathrm{Sh}_!(\mathrm{Aff}_{\mathbb{Z}_\square}, \mathrm{Ani})$ is the ∞-category of solid stacks.

For us, for de Rham stack purposes, we are interested in applying to $\mathrm{Aff}^b_{R_\square}$.

> **Definition**
> $\mathrm{Sh}_!(\mathrm{Aff}^b_{R_\square}, \mathrm{Ani})$ is the ∞-category of *Tate stacks*.

Question from Akhil : We can make a comparison by using the reflector $A \mapsto A^b$?

Answer from audience : Yes.

10 Cartier duality for vector bundles I (Mingjia Zhang)

Motivation : $\mathbb{G}^{\mathrm{alg}}_{a,\mathrm{dR}} = \mathbb{G}_a/\widehat{\mathbb{G}_a}$ over characteristic zero. It will turn out that $\mathbb{G}_{a,\mathrm{dR},R_\square} = \mathbb{G}_a/\mathbb{G}_a^\dagger$. Want to study $D(\mathbb{G}^{\mathrm{alg}}_{a,\mathrm{dR}})$ and relate to D-modules. Need to understand

$$D(B\widehat{\mathbb{G}_a}) \xrightarrow{\sim} D(\mathbb{G}_a)$$

$$D(\widehat{\mathbb{G}_a}) \xrightarrow{\sim} D(B\mathbb{G}_a)$$

and their analytic analogue

$$D(B\mathbb{G}_a^\dagger) \xrightarrow{\sim} D(\mathbb{G}_a^{\mathrm{an}})$$

$$D(\mathbb{G}_a^\dagger) \xrightarrow{\sim} D(B\mathbb{G}_a^{\mathrm{an}})$$

This is called Cartier duality. [1]

We will work in even more restrictive category of solid stacks.

> **Definition**
> $C := \mathrm{Sh}_D(\mathrm{Aff}_{\mathbb{Z}_\square}, \mathrm{Ani})$ is the ∞-category of sheaves w.r.t. the D-topology on $\mathrm{Aff}_{\mathbb{Z}_\square}$. Objects are called *solid D-stacks*.

> **Definition**
> Let X be a solid D-stack. A rank d algebraic vector bundle on X is $F \in D(X)$ that is D-locally free rank d over 1_X.

> **Definition**
> $[*/\mathrm{GL}_d] : C^{\mathrm{op}} \to \mathrm{Ani}$ sends X to the ∞-groupoid of rank d algebraic vector bundles on X. [a]
>
> The identity of $[*/\mathrm{GL}_d]$ corresponds to a rank d algebraic vector bundle $\mathrm{St} \in D([*/\mathrm{GL}_d])$. Then for

---

[1] "Not explicitly stated" is a nice way of saying "skipped".

> $f : X \to [*/\operatorname{GL}_d]$.
>
> ---
>
> [a] Question from audience : is this the quotient in the topos by an actual GL_d?
> Answer from Zhang : yes, GL_d can be defined in usual way over $\mathbb{Z}_\square$.

Question from Niziol : There are lots of D-covers. Doesn't this mean there are many vector bundles?

> **Definition**
> For X solid D-stack $F \in [*/\operatorname{GL}_d](X)$,
>
> $$\mathbb{V}(F) := \underline{\operatorname{AnSpec}}_X \operatorname{Sym}^\bullet_X F^\vee$$
>
> with induced analytic structure from X.
>
> $$\widehat{\mathbb{V}(F)} := \varinjlim_n \underline{\operatorname{AnSpec}}_X \operatorname{Sym}^{\le n}_X F^\vee$$
>
> [a]
>
> ---
>
> [a] where $\operatorname{Sym}^{\le n}_X F^\vee$ means the algebra quotient of $\operatorname{Sym}^\bullet_X F^\vee$ by $(F)^{n+1}$.

Remark.

$$\underline{\operatorname{Hom}}_X(\operatorname{Sym}^n_X F^\vee, 1) = \Gamma^n(F)$$

where $\Gamma^\bullet_X F$ is the divided power algebra. This has 1_X-basis given by $x^{\underline{\alpha}}/n!$ ranging over $x^{\underline{\alpha}} \in \operatorname{Sym}^n F$. This implies

$$\underline{\operatorname{Hom}}_X(\operatorname{Sym}^\bullet_X F^\vee, 1_X) \simeq \widehat{\Gamma^\bullet_X(F)}$$

as co-algebras. $(\Delta : F \to F \otimes F)$ We will then use characteristic zero for

$$\Gamma^n(F) = \operatorname{Sym}^n F$$

so we get a simplification.

> **Proposition**
> Let X be a solid D-stack and $F \in [*/\operatorname{GL}_d](X)$.
>
> 1. $\mathbb{V}(F) \xrightarrow{g} X$ is weakly cohomologically proper
>
> 2. $\widehat{\mathbb{V}(F)} \xrightarrow{f} X$ is cohomologically smooth and there exists an isomorphism
>
> $$f^! 1_X \xrightarrow{\sim} f^* \bigwedge^d F^\vee[d]$$
>
> $$f_! 1_{\widehat{\mathbb{V}(F)}} \xrightarrow{\sim} \bigwedge^d F \otimes \Gamma^\bullet_X F[-d]$$

Question : these isomorphisms are functorial in F? Answer : This is part of the issue in the incomplete proof.

Proof. (1) By construction, g is proper. (2) The properties we want to prove satisfies base change. So WLOG $X = [*/\operatorname{GL}_d]$ and $F = \mathrm{St}$. Cohomological smoothness = $1 \in D(\mathbb{V}(\mathrm{St}))$ is suave w.r.t. $f : \mathbb{V}(\mathrm{St}) \to [*/\operatorname{GL}_d]$ and ω_f is invertible. This be checked $*$-locally on target. So we pullback along $* \to [*/\operatorname{GL}_d]$. [1]

This reduces to the cohomological smoothness of $\widehat{\mathbb{G}_a}^d \to *$. This factors

$$\widehat{\mathbb{G}_a}^d \to \mathbb{P}^d_{\mathbb{Z}_\square} \to *$$

where the first is an open immersion and the latter is cohomologically smooth.

Why is the first open? Take dual basis T_i, $1 \leq i \leq d$ of St.

$$\widehat{\mathbb{G}_a}^d \xrightarrow{j} \mathbb{A}^d_{\mathbb{Z}_\square} = \mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z})_\square \xrightarrow{g} *$$

It is the complement of the idempotent dg-algebra D defined as :

$$\bigoplus_{i=1}^{d} D_i \to \cdots \to \bigoplus_{\{j_1,\dots,j_k\}\subseteq\{1,\dots,d\}} D_{\{j_1,\dots,j_k\}} \to \cdots \to D_{\{1,\dots,d\}}$$

where

$$D_i := (\mathbb{Z}[\underline{\mathrm{T}}, 1/T_i], \mathbb{Z})_\square$$

$$D_{\{j_1,\dots,j_k\}} = \bigoplus_{r=1,\mathbb{Z}[\underline{T}]}^{k} D_j$$

Geometrically $\bigcup_i \mathrm{AnSpec}D_i$.

For $f_! 1_{\widehat{\mathbb{V}}}$ note that

$$(gj)_! 1_{\widehat{\mathbb{G}_a}^d} = g_* j_! 1_{\widehat{\mathbb{G}_a}^d} = \mathrm{Fib}(\mathbb{Z}[\underline{T}] \to D) = \frac{\mathbb{Z}[T_1^{\pm 1}, \dots, T_d^{\pm 1}]}{\bigoplus_{i=1}^d \mathbb{Z}[T_i, T_{j\neq i}^{\pm 1}][-d]} = (T_1 \cdots T_d)^{-1}\mathbb{Z}[T_1^{-1}, \dots, T_d^{-1}][-d]$$

Assuming the Cech complex used is GL_d equivariant, this implies

$$f_! 1_{\widehat{\mathbb{V}(\mathrm{St})}} \xrightarrow{\sim} \bigwedge^d \mathrm{St} \otimes \mathrm{Sym}^\bullet \mathrm{St}[-d]$$

assuming X is over $\mathbb{Q}$. This was the original claim in the Juan's paper. However this is false.

Here's the fix for $f^! 1_X$. Note that

$$f_* : D(\widehat{\mathbb{V}(\mathrm{St})}) \to D(X)$$

exhibits the source as the $(T_1, \cdots, T_d)$-adically complete objects in $\mathrm{Mod}_{f_* 1_{\mathbb{V}(\mathrm{St})}}(D(X))$. So we can compute via

$$f_* f^! 1_X = \underline{\mathrm{Hom}}_X(f_! 1_{\widehat{\mathbb{V}(\mathrm{St})}}, 1_X) = \bigwedge^d \mathrm{St}^\vee \otimes \widehat{\mathrm{Sym}^\bullet \mathrm{St}^\vee}[d]$$

---

[1] Question : The definition of vector bundles shows this is an effective epimorphism in solid D-stacks. But how do we know this is satisfies universal $*$-descent?

Answer : It should be part of the construction that $D(\mathcal{X})$ for a solid D-stack $\mathcal{X}$ is left Kan extended from representables. Then D will send colimits in $\mathrm{Sh}_D(\mathrm{Aff}_{\mathbb{Z}_\square}, \mathrm{Ani})$ to limits in Pr^L_{st}. This implies any effective epimorphism satisfies $*$-descent. Then the fact that effective epimorphisms are universal in topoi implies effective epimorphisms satisfy universal $*$-descent.

Then $f^! 1_X \simeq f^* \bigwedge^d \mathrm{St}^\vee$.

This doesn't solve the problem for $f_! 1_{\widehat{\mathbb{V}(\mathrm{St})}}$. *Convincing proposal* : construct a GL_d equivariant pairing

$$\frac{\mathbb{Z}[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]}{\bigoplus_{i=1}^d \mathbb{Z}[T_i, T_{j\neq i}^{\pm 1}][-d]} \otimes (T_1 \cdots T_d)\mathbb{Z}[\![T_1, \cdots, T_d]\!] \to \mathbb{Z}$$

identifying the left factor with

$$\mathrm{Hom}_{\mathrm{cts}}((T_1 \cdots T_d)\mathbb{Z}[\![T_1, \cdots, T_d]\!], \mathbb{Z})$$

Here $(T_1 \cdots T_d)\mathbb{Z}[\![T_1, \cdots, T_d]\!] \simeq \bigwedge^d \mathrm{St}^\vee \otimes \widehat{\mathrm{Sym}^\bullet \mathrm{St}}^\vee$. The pairing should be $f \otimes g \mapsto \mathrm{const}(fg)$ the constant coefficient.

$\square$

11 Cotangent complex , solid smooth / étale maps (Teruhisa Koshikawa)

Solid smooth is a separate notion to cohomologically smooth, but we will see the relation. This section is roughly based on Lurie's thesis, where Lurie defines cotangent complex for animated rings.

We will work with $\mathrm{PSh}(\mathrm{AnRing}^{\mathrm{op}}, \mathrm{Ani})$. One can forget about 6-functor formalism for this talk. Define $D(X) := \varprojlim_{\mathrm{AnSpec}A \to X} D(A)$ by right Kan extension.

> **Definition**
>
> For $X \in \mathrm{PSh}(\mathrm{AnRing}^{\mathrm{op}}, \mathrm{Ani})$ and $M \in D(X)$, M locally almost connective when for all $x : \mathrm{AnSpec}A \to X$, $x^* M \in D^{\leq 0}(X)$. (almost connective)

For $M \in D^{\leq 0}(A)$, there is a trivial square-zero extension $A \oplus M$,[1] determined by underlying condensed ring $\underline{A} \otimes M$. There are maps $A \to A \oplus M \to A$ which compose the identity.

Fact (Clausen–Scholze) : For $\underline{\tilde{A}} \to \underline{A}$ animated condensed rings and $\ker(\pi_0(\underline{\tilde{A}}) \to \pi_0(\underline{A}))$ is nilpotent ideal, then analytic ring structures on $\underline{\tilde{A}}$ are equivalent to analytic ring structures on $\underline{A}$.

> **Definition**
>
> For X a prestack on animated condensed rings and $\mathbb{L}_X \in D(X)$ locally almost connective, then we say $\mathbb{L}_X$ is an *almost cotangent complex* when for all $x : \mathrm{AnSpec}A \to X$ and $M \in D(A)$,
>
> $$\mathrm{Map}_{D(A)}(x^* \mathbb{L}_X, M) \simeq X(A \oplus M) \times_{X(A)} \{x\}$$
>
> For $f : X \to Y$ a morphism of prestacks over animated analytic rings and $\mathbb{L}_{X/Y}$ locally almost connective, then it is called a relative cotangent complex when for all $x : \mathrm{AnSpec}A \to X$ and $M \in D(A)$,
>
> $$\mathrm{Map}_{D(A)}(x^* \mathbb{L}_{X/Y}, M) \simeq X(A \oplus M) \times_{X(A) \times_{Y(A)} Y(A \oplus M)} \{x\}$$
>
> Intuitively it classifies commuting diagrams

---

[1] I'm guessing one can define this as $\mathrm{Sym}_A^\bullet M/(M)^2$.

$$\begin{tikzcd}
\operatorname{AnSpec} A \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{AnSpec}(A \oplus M) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Proposition – Standard properties 1. $\mathbb{L}_{\varinjlim_i X_i} \simeq \varinjlim_i \mathbb{L}_{X_i}$

2. base change of relative cotangent complexes
3. exact triangle from a triple $X \to Y \to S$.

Proposition

If $f : A \to B$ in animated analytic rings, then $\mathrm{AnSpec}B \to \mathrm{AnSpec}A$ admits a relative cotangent complex given by $B \otimes_{\underline{B}} \mathbb{L}_{\underline{B}/\underline{A}}$

Proof. Reduce to absolute case. For $A \to C$ and $M \in D^{\leq 0}(C)$, then by the fact about analytic ring structures uniquely lifting along nilpotent extensions,

$$\mathrm{Map}_{\mathrm{AnRing}/C}(A, C \oplus M) \simeq \mathrm{Map}_{\mathrm{Cond(AniRing)}/\underline{C}}(\underline{A}, \underline{C} \oplus M) \simeq \mathrm{Map}_{D(\underline{A})}(\mathbb{L}_{\underline{A}}, M) \simeq \mathrm{Map}_{D(A)}(A \otimes_{\underline{A}} \mathbb{L}_{\underline{A}}, M)$$

where $\mathbb{L}_{\underline{A}}$ is the cotangent complex for animated condensed rings. □

Remark. For $B \leftarrow A \to C$ in AnRing, then base change gives

$$\mathbb{L}_{C/A} \otimes_C (C \otimes_A B) \simeq \mathbb{L}_{C \otimes_A B/B}$$

Intuitively, one wants to cancel the C to get

$$\mathbb{L}_{C/A} \otimes_A B$$

However this is not necessarily because this may not land in $D(C)$. This is true under extra assumption of $A \to B$ being *steady*, which we will not discuss further.

Proposition – Criterion for equivalence of analytic rings

For $A \to B$ in AnRing, then it is an equivalence iff it induces $\pi_0 A \simeq \pi_0 B$ and $\mathbb{L}_{B/A} = 0$. [Cam24b, p. 3.4.8]

Proof. STS $\underline{A} \to \underline{B}$ is an equivalence in $D(A)$. Take the cone K in $D(A)$. Suppose $K \neq 0$. Then take minimum n such that $\pi_n K \neq 0$. There is a morphism

$$K \otimes_{\underline{A}} \underline{B} \to \mathbb{L}_{\underline{B}/\underline{A}}$$

which is an isomorphism on π_n. See Lurie's thesis. This implies $K \otimes_A B \to \mathbb{L}_{B/A}$ is an isomorphism on π_n. But the latter is zero by assumption. This implies

$$0 = \pi_n(K \otimes_A B) \simeq \pi_n K \otimes_{\pi_0 A} \pi_0 B = \pi_n K$$

which is a contradiction. □

Example : Let (A, A^+) be discrete classical Tate-Huber pair. Then $\mathbb{L}_{(A,A^+)_\square} \simeq \mathbb{L}_A$. In particular, $\mathbb{L}_{(\mathbb{Z}[T],\mathbb{Z})_\square/\mathbb{Z}_\square} = \mathbb{Z}[T]dT$.

Example : Let $(A, A^+) \to (B, B^+)$ be a morphism of classical Tate-Huber pair. Assume $(A, A^+)_\square \to (B, B^+)_\square$ is steady. Then

$$\pi_0 \mathbb{L}_{(B,B^+)_\square/(A,A^+)_\square} \simeq I/I^2$$

where I is the kernel of

$$\pi_0((B, B^+)_\square \otimes_{(A,A^+)_\square} (B, B^+)_\square) \to (B, B^+)_\square$$

Under finiteness conditions, this gives Huber's Kahler differentials.

Example : For $A \to C$ idempotent,

$$\mathbb{L}_{C/A} \simeq (C \otimes_A C) \otimes_C \mathbb{L}_{C/A} \simeq \mathbb{L}_{C \otimes_A C/C} \simeq \mathbb{L}_{C/C} \simeq 0$$

> **Definition**
>
> (Lurie) For X a prestack on analytic rings, we call it *infinitesimally cohesive* when for any
>
> $$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"'] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$
>
> where $\pi_0 f$ is surjective and $\ker(\pi_0 f)$ is nilpotent, then we have
>
> $$\begin{tikzcd} X(\tilde{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(A) \arrow[d, "f"'] \\ X(\tilde{B}) \arrow[r] & X(B) \end{tikzcd}$$

Remark. A general square zero extension $\tilde{A} \to A$ by a module M is given by

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

> **Definition**
>
> X is *nil-complete* when $X(A) \simeq \varprojlim_n F(\tau^{\geq n} A)$.
>
> A morphism of prestacks on analytic rings $X \to Y$ is called infinitesimally cohesive when for all $x : \mathrm{AnSpec} A \to Y$, the fiber $X \times_Y \mathrm{AnSpec} A$ is infinitesimally cohesive. Similarly for nil-complete.

> Finally, $X \to Y$ is *formally smooth* when it is infinitesimally cohesive, nil-complete, and furthermore $\mathbb{L}_{X/Y}$ exists and is dual of a connective perfect complex.

Now the solid case.

> **Definition**
> Let $f : A \to B$ be a morphism of solid affinoids.
>
> 1. we say it is solid finitely presented when B belongs to the smallest category containing $A[T]_\square = A \otimes_{\mathbb{Z}_\square} \mathbb{Z}[T]_\square$ and stable under finite colimits.
> 2. we say it is solid smooth / étale when it is formally smooth / étale and analytic-locally on B, $A \to B$ is solid finitely presented.
> 3. A standard solid smooth morphism is $A \to B$ where $B = A[T_1, \cdots, T_d]_\square/^L(f_1, \cdots, f_k)$ with $f_1, \cdots, f_r \in \pi_0 A[T_1, \cdots, T_d]$ and Jacobian has maximal rank (which is k).

Example : For A static, if $B := A[T_1, \cdots, T_d]_\square/(f_1, \cdots, f_r)$ is static then $A \to B$ is solid finitely presented.

12 Derived rigid geometry I (Naoki Imai)

These two talks will be about solid étale and solid smooth morphisms.

Goal of first talk : Solid smooth implies cohomologically smooth.

Goal of second talk : Serre duality for solid smooth morphism and solid smooth implies locally †-formally smooth.

> **Definition**
> Let $f : X \to Y$ be a morphism of derived Tate adic spaces.
>
> 1. f affinoid := there exists an affinoid analytic cover $(U_i)_i$ of Y such that $V_i := U_i \times_Y X$ are affinoid.
>
> f strictly affinoid := affinoid and the analytic structure of V_i is induced from U_i.
> 2. f Zariski closed immersion := strictly affinoid and $\mathcal{O}(U_i) \to \mathcal{O}(V_i)$ is surjective on π_0.
> 3. f locally solid finitely presented := analytic locally on X and Y, it is solid finitely presented of bounded affinoid rings.
>
>   solid finite presented f := qcqs and locally solid finitely presented.
> 4. f solid smooth (resp. étale) := analytic locally on X and Y, it is solid smooth (resp. étale) morphism of bounded affinoid rings.

> **Proposition**
> Let $f : X \to Y$ be a morphism of derived Tate adic spaces which is a Zariski closed immersion, solid finitely presented and $\mathcal{O}_X$ is a perfect in $\mathcal{O}_Y$-module. Then
>
> 1. $\mathcal{O}_X = 1_X \in D(X)$ is f-smooth

2. Assume $N^{\vee}_{X/Y} := \mathbb{L}_{X/Y}[-1]$ is locally free. Then $f^!1_Y = (\bigwedge^d N_{X/Y})[-d]$ where $N_{X/Y}$ is the $\mathcal{O}_X$-dual of $N^{\vee}_{X/Y}$ and d is the rank of $N_{X/Y}$.

   In particular, f is cohomologically smooth. [a]

---

[a] Because we don't know derived Tate adic spaces are solid D-stacks, we are secretly using 6-functor formalism on stacks in the analytic topology, rather than for stacks in D-topology.

Proof. (1) For $F \in D(X)$ and $G \in D(Y)$,

$$\mathrm{Hom}_Y(f_!F, G) \simeq \mathrm{Hom}_Y(F, G) \simeq \mathrm{Hom}_X(F, \underline{\mathrm{Hom}}_Y(\mathcal{O}_X, G))$$

where the first uses surjective on π_0, and second uses strictly affinoid. Hence $f^!G \simeq \underline{\mathrm{Hom}}_Y(\mathcal{O}_X, G) \simeq \mathcal{O}_X^{\vee} \otimes_Y G$. Using this, we can check that for pullback squares :

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

we have

$$D_{f'}(1_{X'}) \otimes f^*G \simeq f'^!G$$
$$g'^*f^!G \simeq f'^!g^*G$$

where $D_{f'}(\_) := \underline{\mathrm{Hom}}(\_, f'^!1_Y)$. This implies 1_X is f-smooth.

(2) By considering the relative diagonal Δ_f of f, we have

$$f^!1_Y \simeq \Delta_f^*\pi_1^*f^!1_Y = \Delta_f^*\pi_2^!1_X$$

where $\pi_i : X \times_Y X \to X$ are the projections. Now

$$\pi_2^!1_X \simeq \underline{\mathrm{Hom}}_X(1_{X\times_Y X}, 1_X) \simeq \underline{\mathrm{Hom}}_X(\bigoplus_{i=0}\bigwedge^i N^{\vee}_{X/Y}[i], 1_X)$$
$$\simeq \bigoplus_{i=0}^{d}\bigwedge^i N_{X/Y}[-i] \simeq 1_{X\times_Y X} \otimes_X \bigwedge^d N_{X/Y}[-d] \simeq \pi_2^*\left(\bigwedge^d N_{X/Y}[-d]\right)$$

Hence $f^!1_Y = \bigwedge^d N_{X/Y}[-d]$. □

Proposition

Assume X, Y are derived Tate adic spaces solid smooth over a base derived Tate adic space S, and $f : X \to Y$ a morphism over S which is Zariski closed immersion. Further assume 1_X is a perfect $\mathcal{O}_Y$-module. Then $\mathbb{L}_{X/Y}[-1]$ is locally free.[a] In particular, f is cohomologically smooth.

---

[a] To be precise, this means there is an analytic cover of X such that on the opens U in this cover, $\mathbb{L}_{X/Y}[-1]\big|_U = H^0(\mathbb{L}_{X/Y}[-1]\big|_U)$ using the t-structure on $D(U)$. Note that this does *not* imply that $\mathbb{L}_{X/Y}[-1]\big|_{U\cap V} = H^0(\mathbb{L}_{X/Y}[-1]\big|_{U\cap V})$

> because localisations in analytic geoemtry is not necessarily flat.
>
> > *Lemma. Let $B \to C$ be a standard solid étale map of bounded affinoid rings. Then $C \otimes_B C \to C$ gives an open immersion of solid affinoid spaces.*
> >
> > *Proof.* **There is dissatisfaction with whether standard solid smooth is defined using polynomial algebras or Tate algebras. Couldn't keep up with proof so see [Cam24b, Lemma 3.6.11].** ■
>
> > *Lemma.* $f : \mathrm{AnSpec}\mathbb{Z}[T]_\square \to \mathrm{AnSpec}\mathbb{Z}_\square$ *is cohomologically smooth.*
> >
> > *Proof.* Not very interesting. See [Cam24b, Lemma 3.6.12] ■

> **Proposition**
> Let $f : X \to S$ be a solid smooth (resp. étale) morphism of derived Tate adic spaces. Then f is cohomologically smooth (resp. cohomologically étale).

Proof. The point of checking Zariski closed immersions are cohomologically smooth is that analytic-locally, solid smoothness allows factoring $X \to \mathrm{AnSpec}\mathcal{O}_S[T_1, \cdots, T_d]_\square \to S$ where first is closed immersion and second is projection. ☐

13 Derived rigid geometry II (Naoki Imai)

> **Proposition – Serre duality**
> Let $f : X \to S$ be a solid smooth map of derived Tate adic spaces. Then $f^! 1_S \simeq \Omega^d_{X/S}[d]$ where d is the relative dimension of f and $\Omega^d_{X/S} := \bigwedge^d \mathbb{L}_{X/S}$.

Proof. Again, we use the relative diagonal Δ_f. We have that Δ_f is analytic-locally Zariski closed immersion such that there exists an analytic neighbourhood U of Δ_f where $1_X \in D(U)$ is perfect. Let $\pi_i : X \times_S X \to X$ be the projections. Then using smooth base change

$$f^! 1_S \simeq \Delta_f^* \pi_1^* f^! 1_S \simeq \Delta_f^* \pi_2^! 1_X$$

Assume for a moment that there exists a section $\iota : S \to X$ such that $\mathcal{O}_S$ is a perfect $\mathcal{O}_U$-module for some analytic open neighbourhood $U \subseteq X$ of ιS. Then $\iota^* f^! 1_S = \iota^* \Omega^d_{X/S}[d]$. By deformation to the normal cone, we can construct

$$\begin{tikzcd}
\tilde{X} \arrow[r, shift right, "\tilde{f}"'] \arrow[d] & \tilde{S} := \mathbb{P}^1_S \arrow[l, shift right, "\tilde{\iota}"'] \\
\mathbb{P}^1_X &
\end{tikzcd}$$

where $\tilde{\iota}$ is a section, and

1. over $\mathbb{P}^1 \setminus \{0\}$, $\tilde{\iota}$ is the base change of ι.
2. over $\{0\}$, $\tilde{\iota}$ is the zero section of $N^{\mathrm{an}}_{X/S} \to S$.

Let $\pi : \mathbb{P}^1_S \to S$ be the projection. Then π^* is fully faithful. This follows from the projection formula, proper base change, and the computation that $\pi_* 1_{\mathbb{P}^1_S} = \mathbb{Z}$.

Claim $\tilde{\pi}_* \tilde{f}^! \mathcal{O}_{\tilde{S}}(d)$ belongs to the essential image of π^*. ...

Not very interesting. [Cam24b, Theorem 3.6.15] ☐

Now we move to the topic of †-formally smooth.

- Goal : for smooth rigid spaces X, we want $X \to X^{\mathrm{an}}_{\mathrm{dR}}$ to be an effective epimorphism of Tate stacks.

So we need to know lifting against nilpotent extensions implies lifting against †-nilpotent extension. This is the point of †-formal smoothness.

> **Definition**
>
> Let $A \in \mathrm{AffRing}^b_{R_\square}$. A †-nilpotent ideal of A is full A-module I contained in $\mathrm{Nil}^\dagger(A)$.
>
> Let $T : X \to Y$ in prestacks over bounded affinoids. We say T is †-formally smooth / étale when it is formally smooth / étale and for all $A \in \mathrm{AffRing}^b_{R_\square}$ and †-nilpotent ideal I of A, the morphism of anima
>
> $$X(A) \to X(A/I) \times_{Y(A/I)} Y(A)$$
>
> is surjective (resp. equivalence).[a]
>
> ---
>
> [a] Imai says surjective means surjective on homotopy groups. But shouldn't it be effective epimorphism of anima, equivalently surjective on π_0?

> **Proposition**
>
> A standard solid smooth (resp. étale) morphism of bounded affinoids is †-formally smooth (resp. étale).

Proof. Note that $R_\square \to R\langle T\rangle_\square$ is †-formally smooth. This follows from [Cam24b, Prop.2.6.16.].[1] We show the étale case. Let $A \to B$ be a standard solid étale morphism. By expressing A as a colimit of algebras generated by elements, WLOG $A = R\langle X_1, \cdots, X_d\rangle_\square\langle \mathbb{N}[S]\rangle$ where S is a profinite set and $B = A\langle T_1, \cdots, T_d\rangle /^{\mathbb{L}} (f_1, \cdots, f_d)$ with determinant of Jacobian invertible. By rescaling, we can assume $|f_i| \leq 1$. Then we consider lifting problems

$$\begin{tikzcd}
D/I & B \arrow[l] \arrow[dl, dashed] \\
D \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

where D is a bounded affinoid ring and $I \subseteq D$ is a †-nilpotent ideal.

---

[1] For A solid affinoid, a morphism $R_\square[T] \to A$ of analytic $R_\square$-algebras extends to $R\langle T\rangle_\square$ iff $R_\square[T] \to A^{\dagger\mathrm{red}}$ extends to $R\langle T\rangle_\square$.

$A \to B$ is 0-truncated implies that the space of solutions is contractible iff $\pi_0 = *$.

etc etc not interesting. [Cam24b, Prop. 3.7.5] □

14 Cartier duality II (MingJa Zhang)

Remark. Analytification is defined as $\_^{\mathrm{an}} : \mathrm{PSh}(\mathrm{Aff}_{R_\square}) \to \mathrm{PSh}(\mathrm{Aff}^b_{R_\square}) \to \mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^b_{R_\square})$ where the first is restriction and second is sheafify in analytic topology.

$\mathbb{G}_a^{\mathrm{an}} \subseteq \mathbb{G}_a^{\mathrm{alg}}$ Kan extended back to $\mathrm{Aff}_{R_\square}$ is the open complement to the idempotent $R[T]$-algebra $R\left\{T^{-1}\right\}^\dagger[T]$.

$$\widehat{\mathbb{G}_a} \subseteq \mathbb{G}_a^\dagger \subseteq \overset{\circ}{\mathbb{G}}{}_a^+ \subseteq \overline{\mathbb{G}_a^+} \subseteq \mathbb{G}_a^{\mathrm{an}} \subseteq \mathbb{G}_a^{\mathrm{alg}}$$

Definition

$\mathrm{GL}_d^{+,\dagger}$ is the analytic spectrum of $R\langle T_{i,j}, T\rangle^\dagger/(\det(T_{i,j})T - 1)$. This comes with a morphism

$$[*/\,\mathrm{GL}_d^{+,\dagger}] \to [*/\,\mathrm{GL}_d^{\mathrm{an}}]$$

An analytic vector bundle F of rank d on X a Tate stack is one arising from pulling back along

$$X \xrightarrow{f} [*/\,\mathrm{GL}_d^{\mathrm{an}}]$$

the analytification of the universal rank d vector bundle on $[*/\,\mathrm{GL}_d]$.

If f factors through $[*/GL_d^{+,\dagger}]$ then we say F has a lattice F^+. We say F is a *unitary overconvergent vector bundle*.

We can define the total spaces.

$$\overline{\mathbb{V}(F^+)} := f^*\mathrm{AnSpec}R\langle T_1, \cdots, T_d\rangle^\dagger$$

$$\mathbb{V}(\overset{\circ}{F}{}^+) := \bigcup_{\varepsilon>0} \pi^\varepsilon \overline{\mathbb{V}(F^+)}$$

$$\mathbb{V}(F)^\dagger := \bigcap_{\varepsilon\to\infty} \pi^\varepsilon \overline{\mathbb{V}(F^+)}$$

$$\mathbb{V}(F)^{\mathrm{an}} := \bigcup_{\varepsilon\to\infty} \pi^\varepsilon \overline{\mathbb{V}(F^+)} \simeq (\mathbb{V}(F)^{\mathrm{alg}})^{\mathrm{an}}$$

The main theorems [Cam24b, pp. 4.2.7, 4.3.8, 4.3.13].

Proposition

For X a solid D-stack over $\mathbb{Q}$ and $\mathcal{F}$ a rank d vector bundle on X in the D-topology,

1. there exists a pairing
$$F : \mathbb{V}(\mathcal{F}) \times_X X/\widehat{\mathbb{V}(\mathcal{F}^\vee)} \to B\mathbb{G}_{m,X}$$
such that $F^*\mathcal{O}(1)$ is an isomorphism in $K_{D,X}(\mathbb{V}(\mathcal{F}) \to X/\widehat{\mathbb{V}(\mathcal{F}^\vee)})$ with inverse
$$F^*(\mathcal{O}(-1)) \otimes_{\mathcal{O}_X} \bigwedge^d \mathcal{F}^\vee[-d]$$
The quotient is over X.
2. there exists a pairing
$$G : \widehat{\mathbb{V}(\mathcal{F}^\vee)} \times_X X/\mathbb{V}(\mathcal{F}) \to B\mathbb{G}_{m,X}$$
such that $G^*\mathcal{O}(1)$ is an isomorphism in $K_{D,X}(\widehat{\mathbb{V}(\mathcal{F}^\vee)} \to X/\mathbb{V}(\mathcal{F}))$ with inverse
$$G^*(\mathcal{O}(-1)) \otimes \bigwedge^d \mathcal{F}[d]$$

In particular, the associated Fourier-Mukai transforms yield equivalence of categories.

There are similar statements for $\mathcal{F} : X \to [*/\operatorname{GL}_d^{\mathrm{an}}]$.

Proof. Key inputs :

1. $\mathbb{V}(\mathcal{F}) \to X$ is weakly cohomologically proper
2. $\widehat{\mathbb{V}(\mathcal{F})} \to X$ cohomologically smooth with $f^!1_X \xrightarrow{\sim} f^* \bigwedge^d \mathcal{F}^\vee[d]$ and $f_!1_{\widehat{\mathbb{V}}} \xrightarrow{\sim} \bigwedge^d \mathcal{F} \otimes_{\mathcal{O}_X} (\operatorname{Sym}^\bullet_X \mathcal{F})[-d]$.
3. $X \to X/\mathbb{V}(\mathcal{F}) \to X$ has first map f descendable D-cover and second map g cohomologically smooth and proper, with $g^!1_X = g^* \bigwedge^d \mathcal{F}[d]$.
4. $X \to X/\widehat{\mathbb{V}(\mathcal{F})} \to X$ has first map f smooth D-cover and second map g cohomologically smooth with $g^!1_X \xrightarrow{\sim} g^* \bigwedge^d \mathcal{F}[-d]$.

How to construct the line bundle on $\mathbb{V}(\mathcal{F}) \times_X X/\widehat{\mathbb{V}(\mathcal{F})}$? Use Barr-Beck-Lurie : For S solid D-stack and $\mathcal{F}$ a rank d vector bundle in D-topology on S,
$$D(S/\widehat{\mathbb{V}(\mathcal{F}^\vee)}) \xrightarrow[\sim]{f^!} \operatorname{Mod}_{f^!f_!1_S}(D(S))$$
where $f^!f_!1_S = f^*\underline{\operatorname{End}}(f_!1_S)$. One can compute this as $\operatorname{Sym}^\bullet_X \mathcal{F}^\vee$. (Probably easiest to dualize and compute that this is the Hopf algebra corresponding to the group $\widehat{\mathbb{V}(\mathcal{F}^\vee)}$.) So a line bundle on $\mathbb{V}(\mathcal{F}) \times_X X/\widehat{\mathbb{V}(\mathcal{F})}$ is equivalent to a line bundle on $\mathbb{V}(\mathcal{F})$ with a action from $\operatorname{Sym}^\bullet_X \mathcal{F}^\vee$. We take $\mathcal{L} := \operatorname{Sym}^\bullet_X \mathcal{F}^\vee$ with multiplication action.

To show the Fourier-Mukai transform associated to F is an equivalence, define
$$M := F^*(\mathcal{O}(-1)) \otimes \bigwedge^d \mathcal{F}^\vee[-d]$$

suffices to construct unit and counit maps

$$\Delta_{V,!}1 \xrightarrow{\sim} M \star L$$

$$L \otimes M \xrightarrow{\sim} \Delta_{X/\widehat{V},!}1$$

Now it's a longish computation, based on formal properties of ! and * pushforward, pullbacks. ☐

Remark. Question from Zhang : from Barr-Beck-Lurie we have

$$D(X/\widehat{\mathbb{V}(\mathcal{F}^\vee)}) \xrightarrow[\sim]{f^!} \mathrm{Mod}_{\mathrm{Sym}^\bullet_X \mathcal{F}^\vee}(D(X)) \simeq D(\mathbb{V}(\mathcal{F}))$$

One can ask whether this is precisely FM^{-1}? The guess is no, there is probably a shift by $(\det \mathcal{F})[d]$.

Another question from Zhang : It is not entirely clear that usual tensor corresponds to convolution? [1]

15 Algebraic de Rham stack (Kazuhiro Ito)

Goal : algebraic D-modules of X. This is not used later for analytic D-modules so we will just make everything static. We let $\mathrm{Aff}^{\geq 0}_{\mathbb{Z}_\square}$ denote the full subcategory of static solid affinoids.

> **Definition**
> Let A be a static condensed ring, $I \subseteq A$ an ideal. We say I is uniformly nilpotent when there exists $n \geq 0$ such that for all $f : S \to I$ where S ext.dis., we have $f(s_1) \cdots f(s_n) = 0$ for any $s_1, \ldots, s_n \in S$.

> **Definition**
> Let $X \in \mathrm{PSh}(\mathrm{Aff}_{\mathbb{Z}_\square})$. Define $X^{\mathrm{alg}}_{\mathrm{dR}} \in \mathrm{PSh}(\mathrm{Aff}_{\mathbb{Z}_\square})$ by sending A to $\varinjlim_{I \subseteq \pi_0 \underline{A}} X(\pi_0(A)/I)$ where I ranges over uniformly nilpotent ideals.

Remark. We can compare to schemes. For presheaves X on affines, define

$$X^{\mathrm{Sch}}_{\mathrm{dR}}(A) := \varinjlim_{I \subseteq A} X(A/I)$$

where I ranges over nilpotent ideals.

For X finitely presented scheme over a field, its functor of points is determined by its restriction to finitely presented algebras. The above formula then simplifies to $X(A_{\mathrm{red}})$.

Let $\mathrm{PSh}\,\mathrm{Aff}_{\mathrm{Sch}} \to \mathrm{PSh}\,\mathrm{Aff}^{0\leq}_{\mathbb{Z}_\square}$, $X \mapsto X_\square$ be left Kan extension along $\mathrm{Ring} \to \mathrm{AffRing}^{0\leq}_{\mathbb{Z}_\square}$ given by $A \mapsto (A, A)_\square$. Then $(X_{\mathrm{dR}})_\square = (X_\square)^{\mathrm{alg}}_{\mathrm{dR}}$.

> **Definition**
> Let $\mathbb{A}^1/\mathbb{G}_m \in \mathrm{PStk}(\mathrm{Aff}_{\mathbb{Z}_\square})$ given by
>
> $$A \mapsto \{\mathcal{O}(-1) \text{ line bundle on } \mathrm{AnSpec} A \text{ with a morphism } \mathcal{O}(-1) \to A\}$$

---

[1] The argument should be the same as for any Fourier-Mukai transform. See proof in any book on Fourier-Mukai transforms.

Let $X \in \mathrm{PStk}(\mathrm{Aff}_{\mathbb{Z}_\square})$. Define $X^{\mathrm{alg}}_{\mathrm{dR}^+} \to \mathbb{A}^1/\mathbb{G}_m$ by

$$X^{\mathrm{alg}}_{\mathrm{dR}^+}(\mathcal{O}(-1) \to A) := \varinjlim_{I \subseteq \pi_0 A} X(\mathrm{Cone}(\tilde{I} \otimes \mathcal{O}(-1) \to A))$$

where I ranges over uniformly nilpotent ideals of $\pi_0 A$ and $\tilde{I} := \underline{A} \times_{\pi_0 \underline{A}} I$. This is the *filtered de Rham prestack of X*.

We also define

$$\begin{tikzcd}
X^{\mathrm{alg}}_{\mathrm{Hodge}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\mathrm{alg}}_{\mathrm{dR}^+} \arrow[d] & X^{\mathrm{alg}}_{\mathrm{dR}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
*/\mathbb{G}_m \arrow[r] & \mathbb{A}^1/\mathbb{G}_m & \mathbb{G}_m/\mathbb{G}_m \arrow[l]
\end{tikzcd}$$

We have the example :

Proposition

For $\mathbb{G}_{a,\square} = \mathrm{AnSpec}\mathbb{Z}[T]_\square$, we have

$$(\mathbb{G}_{a,\square})^{\mathrm{alg}}_{\mathrm{dR}^+} = (\mathbb{G}_{a,\square} \times \mathbb{A}^1/\mathbb{G}_m)/\widehat{\mathbb{G}_a(-1)}$$

where $\widehat{\mathbb{G}_a(-1)} := \mathbb{V}(\widehat{\mathcal{O}(-1)}) \to \mathbb{A}^1/\mathbb{G}_m$.

Proof. For simplicity, work over $\mathbb{G}_m/\mathbb{G}_m$. For A static,

$$\widehat{\mathbb{G}_a}(A) = \mathrm{Nil}(\underline{A}(*)) = \varinjlim_{I \subseteq \underline{A}} I(*)$$

So

$$(\mathbb{G}_{a,\square})^{\mathrm{alg}}_{\mathrm{dR}^+}(A) = \varinjlim_{I \subseteq A} \mathbb{G}_{a,\square}(A/I)$$

Then use surjectivity of

$$\mathbb{G}_{a,\square}(A) \to \mathbb{G}_{a,\square}(A/I)$$

with kernel $I(*)$. This follows from $\mathbb{Z}_\square \to \mathbb{Z}[T]_\square$ being formally smooth and $I \subseteq \underline{A}$ being nilpotent. ☐

Definition

Define algebraic de Rham stacks of derived Tate adic spaces as follows :

$$\begin{tikzcd}
\mathrm{Stk}_{\mathrm{an}}(\mathrm{Aff}^b_{R_\square}) \arrow[r, "\text{LKE}"] & \mathrm{PStk}(\mathrm{Aff}_{R_\square}) \arrow[r, "\substack{\mathrm{alg} \\ -\mathrm{dR}}"] & \mathrm{PStk}(\mathrm{Aff}_{R_\square \otimes \mathbb{Q}}) \arrow[r, "\text{sheafify}"] & \mathrm{Stk}_D(\mathrm{Aff}_{R_\square \otimes \mathbb{Q}}) \\
\text{derived Tate adic spaces} \arrow[u]
\end{tikzcd}$$

Similarly for filtered de Rham stack.

Proposition
Let $X \to Y$ in derived Tate adic spaces which is locally of solid finite presentation.

1. The induced morphism $f : X^{\mathrm{alg}}_{\mathrm{dR}} \to Y^{\mathrm{alg}}_{\mathrm{dR}}$ is !-able and we have $f_!, f_*, f^!, f^*$ between $D(X^{\mathrm{alg}}_{\mathrm{dR}})$ and $D(Y^{\mathrm{alg}}_{\mathrm{dR}})$.
2. If f is solid smooth (resp. étale), then $X^{\mathrm{alg}}_{\mathrm{dR}^+} \to Y^{\mathrm{alg}}_{\mathrm{dR}^+}$ is cohomologically smooth (resp. étale).
3. Assume $X = \mathrm{AnSpec}B$ and $Y = \mathrm{AnSpec}A$. If f is solid smooth and D-cover (for example rational open covering) then $X^{\mathrm{alg}}_{\mathrm{dR}^+} \to Y^{\mathrm{alg}}_{\mathrm{dR}^+}$ satisfies univeral $*$-descent and universal !-descent.

Definition
Let $X \to Y$ in derived Tate adic spaces which is solid smooth of relative dimension d. Then we define the *compactly support algebraic de Rham cohomology* as

$$\mathrm{dR}_c(X/Y) := (f^{\mathrm{alg}}_{\mathrm{dR}^+})_!(f^{\mathrm{alg}}_{\mathrm{dR}^+})^! 1_{Y^{\mathrm{alg}}_{\mathrm{dR}^+}} \in D(Y^{\mathrm{alg}}_{\mathrm{dR}^+})$$

Definition – Filtration of $\mathcal{O}_X$
For every $f : X \to \mathbb{A}^1/\mathbb{G}_m$ corresponding to a $f^*\mathcal{O}(-1) \to 1_X \in D(X)$, we obtain a filtration on any $M \in D(X)$ given by tensoring repeatedly with $f^*\mathcal{O}(-1) \to 1_X$.

Proposition
The natural filtration on $\mathrm{dR}_c(X/Y)$ is complete and the graded pieces are

$$\mathrm{gr}^{-i}(\mathrm{dR}_c(X/Y)) \simeq f_!((\Omega^i)^\vee_{X/Y})(-i)[i]$$

...

16 Analytic de Rham stack (Arthur-César le Bras)

We now restrict over the $R_\square$-algebra $\mathbb{Q}_{p,\square}$, i.e. make all pseudo-uniformizers p.

Definition
Let $X \in \mathrm{PStk}(\mathrm{Aff}^b_{\mathbb{Q}_p})$. Define X_{dR} as the D-sheafification of the functor on $\mathrm{Aff}^b_{\mathbb{Q}_p}$

$$A \mapsto X(A^{\dagger \mathrm{red}})$$

We define the relative de Rham stacks for a morphism $X \to Y$ in $\mathrm{Stk}_D(\mathrm{Aff}^b_{\mathbb{Q}_p})$,

> $$\begin{tikzcd} X_{\mathrm{dR}/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{\mathrm{dR}} \arrow[d] \\ Y \arrow[r] & Y_{\mathrm{dR}} \end{tikzcd}$$
>
> Filtered variant $X^+_{\mathrm{dR}} \to \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}}$. This is defined as the D-sheafification of
>
> $$(\mathcal{O}(-1) \to A) \mapsto X(\mathrm{Cone}(\mathrm{Nil}^\dagger(A) \otimes \mathcal{O}(-1) \to A))$$

Question from last talk : What's the (analytic) ring structure on $\mathrm{Cone}(\mathrm{Nil}^\dagger(A) \otimes \mathcal{O}(-1) \to A)$?

le Bras : Don't entirely sure.

Comment from audience : $\mathrm{Nil}^\dagger(A) \otimes \mathcal{O}(-1)$ should be quasi-ideal of A.

Remark. The D-sheafification is undesireable because it is hard to compute.

In algebraic geometry over a field k of characteristic zero, with stacks in Zariski/étale/fppf topology, then

$$R\Gamma_{\mathrm{fppf}}(\mathrm{Spec}\, R, \widehat{\mathbb{G}_a}) = \mathrm{Nil}(R)[0]$$

This appears in lecture notes by Bhatt. This fails already in case of perfect k of characteristic p. Consider

$$k[T] \to k[T^{1/p}]$$

The Cech nerve

$$k[T^{1/p}] \to k[T^{1/p}] \otimes_{k[T]} k[T^{1/p}]$$

Taking reduction gives

$$k[T^{1/p}] \to k[T^{1/p}]$$

So the equalizer gives $k[T^{1/p}]$ rather than $k[T]$. This shows that for already for $X = \mathbb{A}^1_k$ we need to fppf sheafify.

One fix : we can restrict only to semi-perfect rings. Similarly in the analytic situation, one could define the analytic de Rham stack as a functor not on all bounded analytic rings over $\mathbb{Q}_p$ but only on nilperfectoid rings. (Bounded analytic rings over $\mathbb{Q}_p$ with $\dagger$-reduction gives Tate perfectoid ring over $\mathbb{Q}_p$.) Then one does not need to D-sheafify. Then one would need to compare the analytic de Rham stack in the paper to this one for stacks we care about (rigid spaces, classifying stacks of p-adic Lie groups, etc). One needs to know we can find D-covers by nil-perfectoids.

> **Proposition**
> Let $f : X \to Y$ a morphism of derived Tate adic spaces over $\mathbb{Q}_p$.
>
> 1. If f is $\dagger$-formally étale, the morphism
>
> $$X \times \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}} \to X^+_{\mathrm{dR},Y}$$
>
>   is an isomorphism.

> 2. If f is $\dagger$-formally smooth then
> $$X \times \mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}} \to X^+_{\mathrm{dR},Y}$$
> is an effective epimorphism. And we have
> $$X_{\mathrm{Hodge}/Y} := X^+_{\mathrm{dR}/Y} \times_{\mathbb{A}^{1,\mathrm{an}}/\mathbb{G}_m^{\mathrm{an}}} (*/\mathbb{G}_m^{\mathrm{an}}) \simeq (X \times (*/\mathbb{G}_m^{\mathrm{an}}))/T^{\dagger}_{X/Y}(-1)$$

In particular, for X solid smooth Tate adic space over $\mathbb{Q}_p$, $X \to X_{\mathrm{dR}}$ is an effective epimorphism. But this is not true in general, even for finite type adic spaces (okay for rigid spaces). This is false for $X = \overline{\mathbb{D}} := \mathrm{AnSpec}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p)$. In fact

$$\overline{\mathbb{D}}_{\mathrm{dR}} := \left(\underline{\mathbb{D}}^{\dagger}\right)_{\mathrm{dR}}$$

the latter is analytic de Rham stack of overconvergent unit disk.

Example : $(\mathrm{AnSpec}\mathbb{C}_p)_{\mathrm{dR}} = \mathrm{AnSpec}\overline{\mathbb{Q}_p}$.

Example : $\mathbb{G}_{a,\square} = \mathrm{AnSpec}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ affinoid disk. Then

$$\begin{tikzcd}
\mathbb{G}_{a,\square}/\mathbb{G}_a^{\dagger} \arrow[r] \arrow[d, "\sim"'] & */\mathbb{G}_a^{\dagger} \arrow[d] \\
\mathbb{G}_{a,\square,\mathrm{dR}} \arrow[r, "\text{coh. smooth.}"'] & *
\end{tikzcd}$$

Example : Let $\overline{\mathbb{G}_a}^{\dagger}$ the overconvergent unit disk. Then

$$\begin{tikzcd}
\overline{\mathbb{G}_a}^{\dagger}/\mathbb{G}_a^{\dagger} \arrow[r] \arrow[d, "\sim"'] & */\mathbb{G}_a^{\dagger} \arrow[d] \\
(\overline{\mathbb{G}_a}^{\dagger})_{\mathrm{dR}} \arrow[r, "\text{coh. proper}"'] & *
\end{tikzcd}$$

The key input is the lemma from before : $\mathbb{Q}_p[T] \to A$ extends to $\mathbb{Q}_p\langle T\rangle_\square$ iff it extends to $\mathbb{Q}_p\langle T\rangle_\square \to A^{\dagger\mathrm{red}}$.

> **Proposition – 6-functor formalism for filtered analytic D-modules**
>
> There is a 6-functor formalism on (**le Bras does not say, but presumably**) prestacks on bounded affinoids over $\mathbb{Q}_p$ such that
>
> 1. a morphism of derived Tate adic spaces of locally finite presentation are !-able.
> 2. solid smooth (resp. étale) morphisms induce cohomologically smooth (resp. étale) morphism on analytic de Rham stacks.

An interesting point : We have excision. Let X be derived Tate adic space and $U \subseteq X$ an analytic open. Then $U_{\mathrm{dR}} \to X_{\mathrm{dR}}$ is a categorical open immersion. Assume $Z \subseteq X$ is a Zariski closed immersion with open complement U. Then

$$Z = \bigcap_{Z \subseteq V \text{ open}}$$

Assume there exists a cofinal system of closed neighbourhoods of Z. **There is some confusion about the correct conditions but what we want is** : the complement of U_{dR} in X_{dR} is

$$\varprojlim_V V_{\mathrm{dR}} \simeq (\varprojlim_V V)_{\mathrm{dR}} = ((Z \subseteq X)^{\dagger})_{\mathrm{dR}} = Z_{\mathrm{dR}}$$

where the first is as functors on nilperfectoids over $\mathbb{Q}_p$ and the final is the †-neighbourhood of Z in X.

Example : For $Z \subseteq X$ Zariski closed immersion in X, then

$$((Z \subseteq X)^{\dagger})^{\dagger\mathrm{red}} = Z$$

One can see this locally when Z is defined by $I = (f_1, \ldots, f_r)$. This is classically called Kashiwara's equivalence. Question from audience : does this still work for infinitely many generators? le Bras : Yes, but requires more work.

This also works for $Z =$ compactified disk in $X = \mathbb{P}^1_{\mathbb{Q}_p,\square}$ and U the complementary disk.

Now Poincaré duality : It remains to compute the dualizing sheaf for $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to 1$.

> **Proposition**
> Let $f : X \to Y$ be a solid smooth morphism of derived Tate adic spaces. Then
>
> $$f^!_{\mathrm{dR}} 1 \simeq C_X(T^{\mathrm{an}}_{X/Y})$$
>
> where C_X is pullback along zero section of the dualizing sheaf for the morphism $(T^{\mathrm{an}}_{X/Y})_{\mathrm{dR}/Y} \to X_{\mathrm{dR}/Y}$.

Thus we can reduce to the case of $X =$ analytic vector bundle $\mathbb{V}$ over Y where $X_{\mathrm{dR}/Y} = \mathbb{V}/\mathbb{V}^{\dagger}$.

> **Proposition**
> Let $f : X \to Y$ be a solid smooth morphism of derived Tate adic spaces, relative dimension d. Then
>
> $$f^!_{\mathrm{dR}} 1_{Y_{\mathrm{dR}}} \simeq 1_{X_{\mathrm{dR}}}[2d]$$
>
> $$f^{\mathrm{alg},!}_{\mathrm{dR}} 1_{Y^{\mathrm{alg}}_{\mathrm{dR}}} \simeq 1_{X^{\mathrm{alg}}_{\mathrm{dR}}}$$

The second is unexpected. But perhaps solid algebraic D-modules on analytic rings is not the right thing to consider.

Remark. Poincaré duality for solid algebraic D-modules for schemes over $\mathbb{Q}$ can be proved from the 6-functor formalism for the solid algebraic de Rham stack. Some weird things happen. For $f : X \to Y$ morphism of schemes, f smooth implies $f^{\mathrm{alg}}_{\mathrm{dR}}$ is cohomologically prim. Also f proper implies $f^{\mathrm{alg}}_{\mathrm{dR}}$ is cohomologically étale.

Question from Akhil : how is analytic de Rham stack only depends on diamond?

le Bras : Should be but not entirely clear at what generality.

17 Locally analytic representations (Dospinescu)

(I was struggling to keep up with this talk so again, apologies if you find more mistakes in this section.)

Current state of affairs :

1. Heyer–Mann on smooth theory.
2. Juan and other Rodriguez on locally analytic. There are serious complications beyond the case of compact groups.
3. Continuous theory is complete mess.

Let G be a *compact* p-adic Lie group of dimension d, $\mathfrak{g} := \operatorname{Lie} G$. Work with $D(\mathbb{Q}_p) := D(\mathbb{Q}_{p,\square})$. For any possibly non-commutative condensed ring R, Mod_R denotes the stable ∞-category of modules over it in $D(\mathbb{Q}_p)$.

> **Definition**
>
> $C^c(G, \mathbb{Q}_p)$ is the ring of $\mathbb{Q}_p$-valued continuous functions on G.
>
> $C^{\text{la}}(G, \mathbb{Q}_p)$ same for locally analytic functions.
>
> $C^{\text{sm}}(G, \mathbb{Q}_p)$ locally constant functions.
>
> Define $G^? := \operatorname{AnSpec} C^?(G, \mathbb{Q}_p)$. Analytic structure is induced from $\mathbb{Q}_{p,\square}$.[a]
>
> ---
>
> [a] This should mean that $D(C^?(G, \mathbb{Q}_p)) :=$ modules over $C^?(G, \mathbb{Q}_p)$ in $D(\mathbb{Q}_{p,\square})$.

From $C^{sm} \subseteq C^{\text{la}} \subseteq C^c$ we get

$$*/G \to */G^{\text{la}} \to */G^{sm}$$

> **Proposition** 1. $D(*/G^{\text{la}}) \simeq D(C^{\text{la}}(G, \mathbb{Q}_p)\text{-comodules}) \xrightarrow{(*)} \operatorname{Mod}_{D^{\text{la}}}$ where $D^{\text{la}} := \underline{\operatorname{Hom}}_{\mathbb{Q}_p}(C^{\text{la}}, \mathbb{Q}_p)$ with internal hom in the *heart*, i.e. at abelian level. This is a non-commutative algebra with multiplication from the multiplication of G.
>
> Similarly for smooth and continuous.
>
> 2. For smooth and LA, the $(*)$ is fully faithful and has a colimit preserving right adjoint $V \mapsto V^{\text{la}}, V^{sm}$.
>
> For continuous, this is unknown.
>
> 3. $* \to */G^?$ is universal $*$ and universal !-cover. It is proper.
>
> 4. $*/G^? \to *$ is proper and smooth. Furthermore, for LA or continuous,
>
> $$f^! 1 \simeq (\det g)[d]$$
>
> for smooth case, it is the modulus character in degree zero. (It parameterisees Haar measures on

G.)

5. $G^{\mathrm{la}}_{\mathrm{dR}} = G^{c}_{\mathrm{dR}} = G^{sm}$.

Proof. There are some non-formal inputs to these results.

(Geometry of $G^{?}$)

$$C^{sm} = \varinjlim_{H \text{ open} \leq G} C(G/H, \mathbb{Q}_p)$$

$$C^{c} = \left(\varprojlim_{n} C^{sm}(G, \mathbb{Z}/p^n) \right) [1/p]$$

$$C^{\mathrm{la}} = \varinjlim_{H \text{ open} \leq G} C^{H-\mathrm{an}}(G, \mathbb{Q}_p)$$

where $C^{H\text{-an}}(G, \mathbb{Q}_p)$ is *not* finite dimensional.

$$D^{sm} \simeq \varprojlim_{H} \mathbb{Q}_p[G/H]$$

$$D^{c} = \left(\varprojlim_{H} \mathbb{Z}_p[G/H] \right) [1/p]$$

Question from Anschutz : Are the limits derived? Dospinescu : No. These are the classical rings people have studied for long.

We fix $\mathcal{L} \subseteq \mathfrak{g}$ a $\mathbb{Z}_p$-lattice with $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$ such that $\mathcal{L}$ exponentiates to an open normal uniform pro-p subgroup $G_0 \subseteq G$. Then G_0 has an adic model $\mathbb{G}_0$ which is rigid group isomorphic to closed polydisc. Letting $\mathcal{L}$ vary as $p^h\mathcal{L}$ with $h > 0$, we get open and closed polydisks $\overline{\mathbb{G}_h}, \mathbb{G}_h$ whose $\mathbb{Q}_p$-points $\overline{G}_h, G_h \subseteq G$ give a system of neighbourhoods for identity. Then

$$C^{\mathrm{la}}(G) \simeq \varinjlim_{h} C(G/G_h, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}(\overline{\mathbb{G}_h}) \simeq \varinjlim_{h} C(G/G_h, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}(\mathbb{G}_h)$$

Call $\overline{C_h} := \mathcal{O}(\overline{\mathbb{G}_h}), C_h := \mathcal{O}(\mathbb{G}_h)$. Then $D^{\mathrm{la}} \simeq \varprojlim_h \overline{D_h} \simeq \varprojlim_h D_h$.

Now smooth case. Let $D^{sm}_h := \mathbb{Q}_p[G/G_h]$. Then $D^{sm} = \varprojlim_h D^{sm}_h$. Key property : D^{sm}_h is *compact projective* as D^{sm} modules. (Even a retract of D^{sm}.) This is false for D_h.

Replacement :

1. D^h is idempotent over D^{la}.
2. the trivial rep $\mathbb{Q}_p$ is actually perfect as D^c or D^{la} module. (This is a hard theorem of Lazard, Serre, Kohlhasse.)
3. D^{sm} is perfect as D^{la}-module.
4. This resolution allows computation of

   $$R\mathrm{Hom}_{D^{\mathrm{la}}}(\mathbb{Q}_p, D^{\mathrm{la}}) = R\mathrm{Hom}_{D^c}(\mathbb{Q}_p, D^c) = \mathbb{Q}_p(1/(\det \mathfrak{g}))[-d]$$

   This can be seen as Poincaré duality for Lie groups. We also have (much easier to prove)

   $$R\mathrm{Hom}_{D^{sm}}(\mathbb{Q}_p, D^{sm}) = \mathrm{Hom}_G(C^{sm}(G), \mathbb{Q}_p) = (\text{unimodular character})^{-1}[0]$$

(Note to self : These homs should be internal.)

(1) the first equivalence is Barr-Beck and computation of $G^? \otimes_{\mathbb{Q}_p} \cdots \otimes_{\mathbb{Q}_p} G^?$ as $\mathrm{AnSpec} C^?(G \times \cdots \times G, \mathbb{Q}_p)$.

(Interlude) For ? = LA or smooth, we have an endofunctor

$$\mathrm{Mod}_{D^?} \xrightarrow{\_^{?}} \mathrm{Mod}_{D^?}$$
$$V \mapsto \varinjlim_{h} \underline{\mathrm{Hom}}_{D^?}(D_h, V)$$

each term in the colimit is a D_h-module. Then the Rodriguez's prove :

> **Proposition**
>
> $\_^?$ is idempotent, colimit preserving.
>
> $$\mathrm{Rep}^? \, G := \{V \in \mathrm{Mod}_{D^?} \text{ s.t. } V^? = V\}$$
>
> is the smallest full subcategory stable under colimits and contains all Mod_{D_h}.

This hard outside smooth setting because D_h is not compact in LA case. Nuclearity of $\overline{D_h}$ is key.

Important : you can test whether $V \in \mathrm{Mod}_{D^?}$ is in $\mathrm{Rep}^? \, G$ checking the cohomologies lie in $\mathrm{Rep}^? \, G$.

Also useful : $V^{\mathrm{la}} = \underline{\mathrm{Hom}}_{D^{\mathrm{la}}}(\mathbb{Q}_p, C^{\mathrm{la}} \otimes V)$ commutes with colimits.

We can thus put a t-structure via the one on $\mathrm{Mod}_{D^?}$. Then $\mathrm{Rep}^? \, G = D((\mathrm{Rep}^? \, G)^{\heartsuit})$ where the heart can explicitly be identified with comodules over $C^?$. So we get

$$\mathrm{Rep}^? \, G \xrightarrow{\sim} D(\text{comodules over } C^?) \simeq D(*/G^?)$$

Dospinescu : There should be an easier proof of this, but I can't think of any way.

For properness of $*to*/G^?$, suffices to check after pullback along $* \to */G^?$. So we reduce to checking properness of $G^? \to *$. But $C^?$ has induced analytic structure from $\mathbb{Q}_p$.

Next, is $f : */G^? \to *$ proper? We have *exceptional descent* : it suffices to check $\pi_* 1$ is descendable where $\pi : * \to */G^?$. This follows from existence of finite free resolution of $\mathbb{Q}_p$ by $D^?$-modules.

Is f smooth? Want $1_{*/G^?}$ to be suave. Suave objects are stable under retracts, fibers and cofibers. So we only need to find a resolution of this by suave things.

$$\pi_* 1 = C^? \rightsquigarrow \text{ resolution } 0 \to 1_{*/G^?} \to (\pi_* 1)^{n_1} \to \cdots \to (\pi_* 1)^{n_k} \to 0$$

But $\pi_* 1 = \pi_! 1$ is f-suave since π is proper and 1 is $\mathbb{1}$-suave. So f is suave.

$$f_* f^! 1 = f_* \underline{\mathrm{Hom}}(1, f^! 1) = \underline{\mathrm{Hom}}(f_! 1, 1) = \underline{f_* 1, 1}$$

$f_* 1$ is the group cohomology of G with coefficients in $\mathbb{Q}_p$ which is $R\mathrm{Hom}_{D^?}(\mathbb{Q}_p, D^?)$ computed before. □

So now we have

$$\begin{tikzcd}
D(*/G) \arrow[r, shift right, "\top" description, phantom] \arrow[d, phantom, "\simeq"] & D(*/G^{\mathrm{la}}) \arrow[l, shift right, "\text{forget}"'] \arrow[r, phantom, "\top"] \arrow[d, phantom, "\simeq"] & D(*/G^{sm}) \arrow[l, shift right, "\text{forget}"'] \arrow[d, phantom, "\simeq"] \\
\mathrm{Rep}^{c}G \arrow[r, shift right, "\_^{\mathrm{la}}"'] & \mathrm{Rep}^{\mathrm{la}}G \arrow[l, shift right, "\text{forget}"'] \arrow[r, shift right] & \mathrm{Rep}^{\mathrm{sm}}G \arrow[l, shift right]
\end{tikzcd}$$

Dospinescu : for smooth and LA, we have fully faithful functors to derived categories of actual modules over distribution algebras. But for continuous case, this is still not clear. Also, it would be nice to avoid the hard result of Lazard to have the comparison between quasicoherent sheaves on classifying stacks and actual honest representations of groups.

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